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NEW
CONSTRUCTIONS
IN
GRAPHICAL STATICS.

BY

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ILLUSTRATED BY TEN ENGRAVINGS IN TEXT AND NINE FOLDING PLATES.

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PREFACE.

At the meeting of the American Association for the Advancement of Science, held in August, 1876, at Buffalo, the writer read two papers, entitled respectively, "A New Fundamental Method in Graphical Statics," and, "Certain New Constructions in Graphical Statics." The latter paper furnishes the basis of the following pages.

Most of the problems proposed have, it is thought, never been solved heretofore by graphical methods, though partial solutions have been obtained in certain cases.

The possibility of obtaining a direct and complete solution of the various forms of the stiff arch rib is found to depend upon a theorem not hitherto recognized, as to the manner in which the equilibrium curve due to the applied weights is made to coincide as nearly as possible with the curve of the arch, which itself acts as a partial equilibrium curve. It is the difference in position of these two curves which is the measure of the bending moment in the arch. The solution of the arch is further simplified by showing that it depends upon that of a straight girder of the same cross section.

The theorem above referred to, which may be properly named the "Theorem Respecting the Coincidence of Closing Lines," may be considered to occupy in relation to this subject, a place analogous to "Gauss' Theorem of Least Constraint" in Dynamics, or "Moseley's Theorem of Least Resistance" in Statics, and we may perhaps add to that of "Legendre's Method of Least Squares," in the Theory of Observations.

Those who are acquainted with the intricate formulæ used in the analytic solution of this problem are aware that the actual relations are so covered up by these complications that from them a clear understanding of the manner in which the thrust, moment, and shear depend upon the applied weights is difficult, perhaps impossible. But it is hoped that the graphical investigation, which affords a pictorial representation, so to speak, of these quantities and their relations, may present no such difficulties. And further, the thrust, moment, and shear due to changes of temperature, or any cause which alters the span of the arch, are, it is believed, here for the first time obtained by a graphical process.

A new general theorem is also enunciated, which affords the basis for a direct solution of the flexible arch rib, or suspension cable, and its stiffening truss.

These discussions and constructions have led to a new investigation of the continuous girder in the most general case of variable moment of inertia. This investigation furnishes a complete graphical solution of the problem, and is accompanied by an analytic investigation in which the general formulæ appear for the first time in simple form.

Another problem treated is that of the arch having block-work joints, such as are found in stone or brick arches, a case intermediate between the stiff and the flexible arch. A complete graphical solution of this problem was proposed by Poncelet, which the reader will find given by Woodbury* in the case where the arch and load are symmetrical about the crown. The solution proposed is far simpler, susceptible of greater accuracy, and is not restricted by considerations of symmetry. Woodbury states that the solution given by him is correct for an unsymmetrical arch, but in this he is mistaken.

The graphical construction for the stability of retaining walls is the first one proposed, so far as known, which employs the true thrust in its real direction, as shown by Rankine in his classic investigation of the thrust of homogeneous solids. It is in fact an adaptation of that most useful conception, "Coulomb's Wedge of Maximum Pressure" to the results of Rankine's investigation.

It has been found possible to obtain a complete solution of the dome by employing constructions analogous to those employed for the arch; and in particular it is believed that the dome of masonry is here investigated correctly for the first time, and the proper distinctions pointed out between it and the dome of metal.

Finally, it may perhaps be said with truth, that neither of these problems can be solved with the same generality by analytic processes as by a graphical construction. The analysis almost always demands some kind of law or uniformity in the loading and in the structure sustaining the load, while the graphical method treats all cases without increase of complexity; and especially are the cases of discontinuity, either in the load or structure, difficult by analysis but easy by graphics.

H. T. E.

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* Stability of the Arch, by D. P. Woodbury, page 404. Published by D. Van Nostrand, New York, 1858.

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NEW CONSTRUCTIONS IN GRAPHICAL STATICS.

CHAPTER I.

It is the object of this work to fully discuss the stability of all forms of the arch, flexible or rigid, by means of the equilibrium polygon—the now well recognized instrument for graphical investigation. One or two other constructions of interest may also be added in the sequel. The discussion will pre-suppose an elementary knowledge of the properties of the equilibrium polygon, and its accompanying force polygon, for parallel forces.

As ordinarily used in the discussion of the simple or continuous girder, the equilibrium polygon has an entirely artificial relation to the problem in hand, and the particular horizontal stress assumed is a matter of no consequence; but not so with respect to the arch. As will be seen, there is a special equilibrium polygon appertaining to a given arch and load, and in this particular polygon the horizontal stress is the actual horizontal thrust of the arch. When this thrust has been found in any given case, it permits an immediate determination of all other questions respecting the stresses. This thrust has to be determined differently in arches of different kinds, the method being dependent upon the number, kind, and position of the joints in the arch.

The methods we shall use depend upon our ability to separate the stresses induced by the loading into two parts; one part being sustained in virtue of the reaction of the arch in the same manner as an inverted suspension cable (*i.e.*, as an equilibrated linear arch), and the remainder in virtue of its reaction as a girder. These two ways in which the loading is sustained are to be considered somewhat apart from each other. To this end it appears necessary to restate and discuss, in certain aspects, the well-known equations applicable to elastic girders acted on by vertical pressures due to the load and the resistances of the supports.

Let P represent any one of the various pressures, P_1, P_2, P_n, applied to the girder.

Consider an ideal cross section of the girder at any point O.

Let x=the horizontal distance from O to the force P.

Let R=the radius of curvature of the girder at O.

At the cross section O, the equations just mentioned become:—

Shearing stress, $S=\Sigma\,(P)$

Moment of flexure, $M=\Sigma\,(Px)$

Curvature, $P'=\frac{1}{R}=\frac{M}{EI}$

Total bending, $B=\Sigma(P')=\Sigma\left(\frac{M}{EI}\right)$

Deflection, $D=\Sigma\,(P'x)=\Sigma\left(\frac{Mx}{EI}\right)$

in which E is the modulus of elasticity of the material, and I is the moment of inertia of the girder; and as is well known, the summation is to be extended from the point O to a free end of the girder, or, if not to a free end, the summation expresses the effect only of the quantities included in the summation.

Let a number of points be taken at equal distances along the girder, and let the values of P, S, M, B, D be computed for these points by taking O at these points successively, and also erect ordinates at these points whose lengths are proportional to the quantities computed. First, suppose I is the same at each of the points chosen, then the values of these ordinates may be expressed as follows, if a, b, c, etc., are any real constants whatever :

$$y_p = a\,.\,P \quad . \quad . \quad . \qquad (1)$$

$$y_s = b\,.\,\Sigma(P) \quad . \quad . \qquad (2)$$

$$y_m = c\,.\,\Sigma(Px) = c\,.\,M\,. \qquad (3)$$

$$y_b = d\,.\,\Sigma(M) \quad . \quad . \qquad (4)$$

$$y_d = e\,.\,\Sigma(Mx) \quad . \quad . \qquad (5)$$

If I is not the same at the different cross sections, let $P{=}M \div I$; then the last three equations must be replaced by the following:

$$y_m' = f\,.\,P' \quad . \quad . \qquad (3')$$

$$y_b' = g\,.\,\Sigma(P') \quad . \quad . \qquad (4')$$

$$y_d' = h\,.\,\Sigma(P'x) \quad . \quad . \qquad (5')$$

The ordinates y_m and y_m' are not equal, but can be obtained one from the other when we know the ratio of the moments of inertia at the different cross sections.

Equation (1) expresses the loading, and y_p may be considered to be the depth of some uniform material as earth, shot or masonry constituting the load. Lines joining the extremities of these ordinates will form a polygon, or approximately a curve which is the upper surface of such a load. When the load is uniform the surface is a horizontal line.

For the purposes of our investigation, a distributed load whose upper surface is the polygon or curve, above described, is considered to have the same effect as a series of concentrated loads proportional to the ordinates y_p acting at the assumed points of division. If the points of division be assumed sufficiently near to each other, the assumption is sufficiently accurate.

If a polygon be drawn in a similar manner by joining the extremities of the ordinates y_m computed from equation (3), it is known that this polygon is an equilibrium polygon for the applied weights P, and it can also be constructed directly without computation by the help of a force polygon having some assumed horizontal stress.

Now, it is seen by inspection that equations (3) and (5), or (3′) and (5′), have the same relationship to each other that equations (1) and (3) have. The relationship may be stated thus :—If the ordinates y_m (or y_m') be regarded as the depth of some species of loading, so that the polygonal part of the equilibrium polygon is the surface of such load, then a second equilibrium polygon constructed for this loading will have for its ordinates proportional to y_d. But these last are proportional to the actual deflections of the girder.

Hence a second equilibrium polygon, so constructed, might be called the deflection polygon, as it shows on an exaggerated scale the shape of the neutral axis of the deflected girder.

The first equilibrium polygon having the ordinates y_m may be called the moment polygon.

It may be useful to consider the physical significance of equations (3), (4), (5), or (3′), (4′), (5′).

According to the accepted theory of perfectly elastic material, the sharpness of the curvature of a uniform girder is directly proportional to the moment of the applied forces, and for different girders or different portions of the same girder, it is inversely proportional to the resistance which the girder can afford. Now this resistance varies directly as I varies, hence curvature varies as $M \div I$, which is equation (3) or (3′).

Now curvature, or bending at a point, is expressed by the acute angle between two tangents to the curve at the distance of a unit from each other; and the total bending, *i.e.* the angle between the tangent at O, and that at some distant point A is the sum of all such angles between O and the point A. Hence the total bending is proportional to $\Sigma(M \div I)$, the summation being extended from O to the point A, which is equation (4) or (4′).

Again, if bending occurs at a point distant from O, as A, and the tangent at A be considered as fixed, then O is deflected from this tangent, and the amount of such deflection depends both upon the amount of the bending at A, and upon its distance from O. Hence the deflection from the tangent at A is proportional to $\Sigma\,(Mx \div I)$ which is equation (5) or (5′).

It will be useful to state explicitly several propositions, some of which are implied in the foregoing equations. The importance and applicability of some of them has not, perhaps, been sufficiently recognized in this connection.

Prop. I. Any girder (straight or otherwise) to which vertical forces alone are applied (*i.e.*, there is no horizontal thrust) sustains at any cross-section the stress due to the load, solely by developing one internal resistance equal and opposed to the shearing, and another equal and opposed to the moment of the applied forces.

Prop. II. But any flexible cable or arch with hinge joints can offer no resistance at these joints to the moment of the applied forces, and their moment is sustained by the horizontal thrust developed at the supports and by the tension or compression directly along the cable or arch.

It is well known that the equilibrium polygon receives its name from its being the shape which such a flexible cable, or equilibrated arch, assumes under the action of the forces. In this case we may say for brevity, that the forces are sustained by the cable or arch in virtue of its being an equilibrium polygon.

Prop. III. If an arch not entirely flexible is supported by abutments against which it can exert a thrust having a horizontal component, then the moment due to the forces applied to the arch will be sustained at those points which are not flexible, partly in virtue of its being approximately an equilibrium polygon, and partly in virtue of its resistance as a girder.

It is evident from the nature of the equilibrium polygon that it is possible with any given system of loading to make an arch of such form (viz., that of an equilibrium polygon) as to require no bracing whatever, since in that case there will be no tendency to bend at any point. Also it is evident that any deviation of part of the arch from this equilibrium polygon would need to be braced. As, for example, in case two distant points be joined by a straight girder, it must be braced to take the place of part of the arch. Furthermore, the greater the deviation the greater the bending moment to be sustained in this manner. Hence appears the general truth stated in the proposition.

It will be noticed that the moment called into action, at any point of a straight girder, depends not only on the applied forces which furnish the polygonal part of the equilibrium polygon, but also on the resistance which the girder is capable of sustaining at joints or supports, or the like. For example, if the girder rests freely on its end-supports, the moment of resistance vanishes at the ends, and the "closing line" of the polygon joins the extremities of the polygonal part. If however the ends are fixed horizontally and there are two free (hinge) joints at other points of the girder, the polygonal part will be as before, but the closing line would be drawn so that the moments at those two points vanish. Similarly in every case (though the conditions may be more complicated than in the examples used for illustration) the position of the closing line is fixed by the joints or manner of support of the girders, for these furnish the conditions which the moments (*i. e.*, the ordinates of the equilibrium polygon) must fulfill. For example, in a straight uniform girder without joints and fixed horizontally at the ends, the conditions are evidently these; the total bending vanishes when taken from end to end, and the deflection of one end below the tangent at the other end also vanishes.

Prop. IV. If in any arch that equilibrium polygon (due to the weights) be constructed which has the same horizontal thrust as the arch actually exerts; and if its closing line be drawn from consideration of the conditions imposed by the supports, etc.; and if furthermore the curve of the arch itself be regarded as another equilibrium polygon due to some system of loading not given, and its closing line be also found from the same considerations respecting supports, etc., then, when these two polygons are placed so that these closing lines coincide and their areas partially cover each other, the ordinates intercepted between these two polygons are proportional to the real bending moments acting in the arch.

Suppose that an equilibrium polygon due to the weights be drawn having the same horizontal thrust as the arch. We are in fact unable to do this at the outset as the horizontal thrust is unknown. We only suppose it drawn for the purpose of discussing its properties. Let also the closing line be drawn, which may be done, as will be seen hereafter. Call the area between the closing line and the polygon, A. Draw the closing line of the curve of the arch itself (regarded as an equilibrium polygon) according to the same law, and call the area between this closing line and its curve A''. Further let A' be the area of a polygon whose ordinates represent the actual moments bending the arch, and drawn on the same scale as A and A''. Since the supports etc., must influence the position of the closing line of this polygon in the same manner as that of A, we have by Prop. III not only

$$A=A'+A''$$

which applies to the entire areas, but also

$$y=y'+y''$$

as the relation between the ordinates of these polygons at any of the points of division before mentioned, from which the truth of the proposition appears.

This demonstration in its general form may seem obscure since the conditions imposed by the supports, etc., are quite various, and so cannot be considered in a general demonstration. The obscurity, however, will disappear after the treatment of some particular cases, where we shall take pains to render the truth of the proposition evident. We may, however, make a statement which will possibly put the matter in a clearer light by saying that A'' is a figure easily found, and we, therefore, employ it to assist in the determination of A' which is unknown, and of A which is partially unknown. And we arrive at the peculiar property of A'', that its closing line is found in the same manner as that of A, by noticing that the positions of the closing lines of A and A' are both determined in the same manner by the supports, etc.; for the same law would hold when the rise of the arch is nothing as when it has any other value. But A'' is the difference of A and A'. Hence what is true of A and A' separately is true of their difference A'', the law spoken of being a mere matter of summation.

From this proposition it is also seen that the curve of the arch itself may be regarded as the curved closing line of the polygon whose ordinates are the actual bending moments, and the polygon itself is the polygonal part of the equilibrium polygon due to the weights.

It is believed that Prop. IV contains an important addition to our previous knowledge as to the bending moments in an arch, and that it supplies the basis for the heretofore missing method of obtaining graphically the true equilibrium polygon for the various kinds of arches.

Prop. V. If bending moments M act on a uniform inclined girder at horizontal distances x from O, the amount of the vertical deflection yd will be the same as that of a horizontal girder of the same cross section, and having the same horizontal span, upon which the same moments M act at the same horizontal distances x from O. Also, if bending moments M act as before, the amount of the horizontal deflection, say x_d, will be the same as that of a vertical girder of the same cross section, and having the same height, upon which the same moments M act at the same heights.

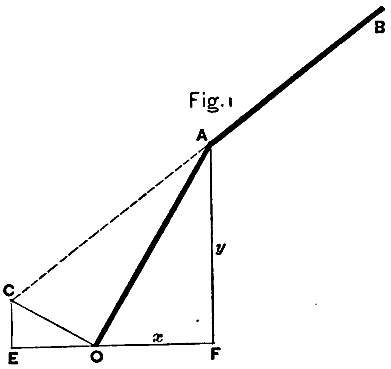

Fig. 1

Let the moment M act at A, producing according to equation (5) the deflection

$$OC = e.M.AO$$

whose vertical and horizontal components are

$$y_d = CE \quad \text{and} \quad x_d = OE$$

For the small deflections occurring in a girder or arch, $AOC = 90°$

$$\therefore\ AO : OF :: OC : CE$$

$$\therefore\ CE = \frac{OC}{AO}.OF = e.M.OF$$

$$\therefore\ y_d = e.Mx$$

$$\text{Also, } AO : AF :: OC : OE$$

$$\therefore\ OE = \frac{OC}{AO}.AF = e.M.AF$$

$$\therefore\ x_d = e.My$$

The same may be proved of any other moments at other points; hence a similar result is true of their sum; which proves the proposition.

It may be thought that the demonstration is deficient in rigor by reason of the assumption that $AOC = 90°$.

Such, however, is not the fact as appears from the analytic investigation of this question by Wm. Bell in his attempted graphical discussion of the arch in Vol. VIII of this Magazine, in which the only approximation employed is that admitted by all authors in assuming that the curvature is exactly proportional to the bending moment.

We might in this proposition substitute $f.M \div I$ for $e.M$, and prove a similar but more general proposition respecting deflections, which the reader can easily enunciate for himself.

Before entering upon the particular discussions and constructions we have in view, a word or two on the general question as to the manner in which the problem of the arch presents itself, will perhaps render apparent the relations between this and certain previous investigations. The problem proposed by Rankine, Yvon-Villarceaux, and other analytic investigators of the arch, has been this:—Given the vertical loading, what must be the form of an arch, and what must be the resistances of the spandrils and abutments, when the weights produce no bending moments whatever? By the solution of this question they obtain the equation and properties of the particular equilibrium polygon which would sustain the given weights. Our graphical process completely solves this question by at once constructing this equilibrium polygon. It may be remarked in this connection, that the analytic process is of too complicated a nature to be effected in any, except a few, of the more simple cases, while the graphical process treats all cases with equal ease.

But the kind of solution just noticed, is a very incomplete solution of the problem presented in actual practice; for, any moving load disturbs the distribution of load for which the arch is the equilibrium polygon, and introduces bending moments. For similar reasons it is necessary to stiffen a suspension bridge. The arch must then be proportioned to resist these moments. Since this is the case, it is of no particular consequence that the form adopted for the arch in any given case, should be such as to entirely avoid bending moments when not under the action of the moving load.

So far as is known to us, it is the universal practice of engineers to assume the form and dimensions, as well as the loading of any arch projected, and next to determine whether the assumed dimensions are consistent with the needful strength and stability. If the assumption is unsuited to the case in hand, the fact will appear by the introduction of excessive bending moments at certain points. The considerations set forth furnish a guide to a new assumption which shall be more suitable, it being necessary to make the form of the arch conform more closely to that of the equilibrium polygon for the given loading.

The question may be regarded as one of economy of material, and ease of construction, analogous to that of the truss bridge. In this latter case, constructors have long since abandoned any idea of making bridges in which the inclination of the ties and posts should be such as to require theoretically the minimum amount of material. Indeed, the amount of material in the case of a theoretic minimum, differs by such an inconsiderable quantity from that in cases in which the ties and posts have a very different inclination, that the attainment of the minimum is of no practical consequence.

Similar considerations applied to the arch, lead us to the conclusion that the form adopted can in every case be composed of segments of one or more circles, and that for the purpose of construction every requirement will then be met as fully as by the more complicated transcendental curves found by the writers previously mentioned. If considerations of an artistic nature render it desirable to adopt segments of parabolas, ellipses or other ovals, it will be a matter of no more consequence than is the particular style of truss adopted by rival bridge builders.

We can also readily treat the problem in an inverse manner, viz :—find the system of loading, of which the assumed curve of the arch is the equilibrium polygon. From this it will be known how to load a given arch so that there shall be no bending moments in it. This, as may be seen, is often a very useful item of information ; for, by leaving open spaces in the masonry of the spandrils, or by properly loading the crown to a small extent, we may frequently render a desirable form entirely stable and practicable.

CHAPTER II.

THE ARCH RIB WITH FIXED ENDS.

LET us take, as the particular case to be treated, that of the St. Louis Bridge, which is a steel arch in the form of the arc of a circle ; having a chord or span of 518 feet and a versed sine or rise of one-tenth the span, *i. e.* 51.8 feet. The arch rib is firmly inserted in the immense skew-backs which form part of the upper portion of the abutments. It will be assumed that the abutments do not yield to either the thrust or weight of the arch and its load, which was also assumed in the published computations upon which the arch was actually constructed. Further, we shall for the present assume the cross section of the rib to have the same moment of inertia, I, at all points, and shall here only consider the stresses induced by an assumed load. The stresses due to changes in the length of the arch itself, due to its being shortened by the loading, and to the variations of temperature, are readily treated by a method similar to the one which will be used in this article, and will be treated in a subsequent chapter.

Let $b_8\, a\, b_8'$ in Fig. 2, be the neutral axis of the arch of which the rise is one-tenth the span. Let $a\, x\, y\, z$ be the area representing the load on the left half of the arch, and $a\, x'\, y'\, z'$ that on the right, so that $yp = a\,.\,P = xy$ on the left, and $yp = x'y'$ on the right.

Divide the span into sixteen equal parts bb_1, bb_1', etc, and consider that the load, which is really uniformly distributed, is applied to the arch at the points a, a_1, a_1', etc., in the verticals through b, b_1, b_1', etc.; so that the equal weights P are applied at each of the points on the left of a and the equal weights $\frac{1}{2}P$ at each point on the right of a, while $\frac{3}{4}P$ is applied at a.

Take b as the pole of a force polygon for these weights, and lay off the weights which are applied at the left of a on the vertical through b_8, viz., $b_8\, w_1 = \frac{1}{2}P =$ the weight coming to a from the left ; $w_1\, w_2 = P =$ the weight applied at a_1 ; $w_2\, w_3 = P =$ the weight applied at a_2, etc. Using b still as the pole, lay off $b_8'\, w_1' = \frac{1}{4}P =$ the weight coming to a from the right; $w_1'\, w_2' = \frac{1}{2}P =$ the weight applied at a_1', etc. This amounts to the same thing as if all the weights were laid off in the same vertical. Part are put at the left and part at the right for convenience of construction. Now draw bw_1 until it intersects the vertical 1 at c_1; then draw $c_1\, c_2 \parallel bw_2$; and $c_2\, c_3 \parallel bw_3$, etc. In the same manner draw bw_1' to c_1'; then $c_1' c_2' \parallel bw_2'$, etc. Then the broken line $bc_1 \ldots c_8$ is the equilibrium polygon due to the weights on the left of a, and $bc_1' \ldots c_8'$ is that due to the weights on the right. Had the polygon been constructed for the uniformly distributed load (not considered as concentrated), on the left we should have a parabola passing through the points $bc_1 \ldots c_8$, and another parabola on the right through $bc_1' \ldots c_8'$. From the properties of this parabola it is easily seen that c_8 must bisect $w_4 w_5$, as c_8' must also bisect $w_4' w_5'$; which fact serves to test the accuracy of our construction. This test is not so simple in cases of more irregular loading.

The equilibrium polygon $c_8 \, b \, c_8'$ is that due to the applied weights, but if these weights act on a straight girder with fixed ends, this manner of support requires that the total bending be zero, when the sum is taken of the bending at the various points along the entire girder; for, the position of the ends does not change under the action of the weights, hence the positive must cancel the negative bending. To express this by our equations:

$$yb=e. \ \Sigma(M)=0 \ \therefore \ \Sigma(M)=0.$$

This is one of two conditions which are to enable us to fix the position of the true closing line $h_8 h_8'$ in this case. The other condition results from the fact that the algebraic sum of all the deflections of this straight girder must be zero if the ends are fixed horizontally.

This is evident from the fact that when one end of a girder is built in, if a tangent be drawn to its neutral axis at that end, the tangent is unmoved whatever deflections may be given to the girder; and if the other end be also fixed, its position with reference to this tangent is likewise unchanged by any deflections which may be given to the girder. To express this by our equations:

$$yd=f. \ \Sigma\,(Mx)=0 \ \therefore \ \Sigma\,(Mx)=0$$

The method of introducing these conditions is due to Mohr. Consider the area included between the straight line $c_8 c_8'$ and the polygon $c_8 \, b \, c_8'$ as some species of plus loading; we wish to find what minus loading will fulfill the above two conditions. Evidently the whole negative loading must be equal numerically to the whole positive loading, if we are to have $\Sigma\,(M)=0$. Next, as the closing line is to be straight, the negative load $c_8 \, c_8' \, h_8 \, h_8'$ may be considered in two parts, viz., the two triangles, $c_8 \, c_8' \, h_8$ and $c_8' \, h_8 \, h_8'$. Let the whole span be trisected at t and t', then the total negative loading may be considered to be applied in the verticals through t and t', since the centers of gravity of the triangles fall in these verticals. Again, the positive loading we shall find it convenient to distribute in this manner: viz., the triangle $c_8 \, b \, c_8'$ applied in the vertical through b, the parabolic area $b \, c_1 \ldots c_8$ in the vertical 4 which contains its center of gravity, and the parabolic area $bc_1' \ldots c_8'$ in 4′.

Now these areas must be reduced to equivalent triangles or rectangles, with a common base, in order that we may compare the loads they represent. Let the common base be half the span: then $bb_0=pp'$ is the positive load due to the triangle $c_8 \, b \, c_8'$; and $\frac{2}{3} \, c_4 \, c_0=pp_1$ and $\frac{2}{3} \, c_4' \, c_0'=p'p_1'$ are the positive loads due to the parabolic areas.

Now assume any point q as a pole for the load line $p_1 \, p_1'$ and find the center of gravity of the positive loading by drawing the equilibrium polygon, whose sides are parallel to the lines of this force polygon: viz., use qp_1 and qp as the 1st and 2nd sides, and make $pq' \parallel qp'$, and $q'q_1 \parallel qp_1'$. The first and last sides intersect at q_1; therefore the center of gravity of the positive loads must lie in the vertical through q_1.

Now the negative loading must have its center of gravity in the same vertical, in order that the condition $\Sigma\,(Mx)=0$ may be satisfied, for it is the numerator of the general expression for finding the center of gravity of the loading. The question then assumes this form: what negative loads must be applied in the verticals through t and t' that their sum may be $p_1 \, p_1'$, and that they may have their center of gravity in the vertical through q_1.

The shortest way to obtain these two segments of $p_1 p_1'$ is to join r and r' which are in the horizontals through p_1 and p_1', and draw an horizontal through q_0, which is the intersection of $r \, r'$ with the vertical through q_1; then $r \, r_1$ and $r' r_1'$ are the required segments of the negative load. For, let $r r_2 = p_1' p_1$ and take r' as the pole of the load $r r_2$; then, since $r_1 q_0 \parallel r_2 r'$ and $q_0 r' \parallel r r'$ we have the equilibrium polygon $r_1 q_0 r'$ fulfilling the required conditions.

Now these two negative loads $r_1 r_2 = r_1' r'$ and $r r_1$, are the required heights of the triangles $c_8 h_8 c_8'$ and $c_8 c_8' h_8'$; therefore lay off $c_8 h_8 = r' r_1'$ and $c_8' h_8' = r r_1$.

The closing line $h_8 h_8'$ can then be drawn, and the moments bending the straight girder will then be proportional to $h_1 c_1$, $h_2 c_2$, etc., the points of inflexion being where the closing line intersects the polygon. If the construction has been correctly made, the area above the closing line is equal to that below, a test easy to apply.

Let us now turn to the consideration of the curve of the arch itself, and treat it as an equilibrium polygon. Since the rise of the arch is such a small fraction of the span, the curve itself is rather flat for our purposes, and we shall therefore multiply its ordinates ab, $a_1 b_1$, etc., by any number convenient for our purpose: in this case, say, by 3. We thereby get a polygon $d_8 d d_8'$ such that $d b = 3\ ab$, $d_1 b_1 = 3\ d_1 b_1$, etc. If a curve be described through $d_8 \ldots d \ldots d_8'$ it will be the arc of an ellipse, of which d is the extremity of the major axis.

If we wish to find the closing line $k_8 k_8'$ of this curve, such that it shall make $\Sigma (Md) = 0$ and $\Sigma (Mdx) = 0$, the same process we have just used is here applicable; but since the curve is symmetrical, the object can be effected more easily. By reason of the symmetry about the vertical through b, the center of gravity of the positive area above the horizontal through b lies in the vertical through b. The center of gravity of the negative area lies there also; hence the negative area is symmetrical about the center vertical; the closing line must then be horizontal. It only remains then to find the height of a rectangle having the same area as the elliptical segment, and having the span for its base. This is done very approximately by taking (in this case where the span is divided into 16 equal segments) $\frac{1}{8}$ the sum of the ordinates $b_1 d_1$, etc.

We thus find the height bk and the horizontal through k is the required closing line.

Before effecting the comparison which we intend to make between the polygons c and d (as we may briefly designate the polygons $c_8 b c_8'$ and $d_8 d d_8'$), let us notice the significance of certain operations which are of use in the construction before us. One of these is the multiplication of the ordinates of the polygon or curve a to obtain those of d. If a was inverted, certain weights might be hung at the points a_1, a_2, etc., such that the curve would be in stable equilibrium, even though there are flexible joints at these points. Equilibrium would still exist in the present upright position under these same applied weights, though it would be unstable. If now, radiating from any point, we draw lines, one parallel to each of the sides aa_1, $a_1 a_2$, aa_1', etc., of the polygon, then any vertical line intersecting this pencil of radiating lines will be cut by it in segments, which represent the relative weights needed to make a their equilibrium polygon. By drawing the vertical line at a proper distance from the pole, its total length, *i. e.*, the total load on the arch can be made of any amount we please. The horizontal line from the pole to this vertical will be the actual horizontal thrust of the arch measured on the same scale as the load. If a like pencil of radiating lines be drawn parallel to the sides of the polygon d and the load be the same as that we had supposed upon the polygon a, it is at once seen that the pole distance for d is one-third of that for a; for, every line in d has three times the rise of the corresponding one in a, and hence with the same rise, only one-third the horizontal span. The increase of ordinates, then, means a decrease of pole distance in the same ratio, and vice versa. As is well known, the product of the pole distance by the ordinate of the equilibrium polygon is the bending moment. This product is not changed by changing the pole distance.

Again, suppose the vertical load-line of a force polygon to remain in a given position, and the pole to be moved vertically to a new position. No vertical or horizontal dimension of the force polygon is affected by this change, neither will any such dimension of the equilibrium polygon corresponding to the new position of the pole be different from that in the polygon corresponding to the first position of the pole; the direction of the closing line, however, is changed. Thus we see that the closing line of any equilibrium polygon can be made to coincide with any line not vertical, and that its ordinates will be unchanged by the operation. It is unnecessary to draw the force polygon to effect this change.

Now to make clear the relationship between the polygons c and d, let us suppose, for the instant, that the polygon e has been drawn by some means as yet unknown, so that its ordinates from d, viz., $e_1 d_1 = y_1$, $e_2 d_2 = y_2$, etc., are proportional to the actual moments Me which tend to bend the arch.

The conditions which then hold respecting these moments Me, are three:—

$$\Sigma(Me)=0,\ \Sigma(Mex)=0,\ \Sigma(Mey)=0.$$

The first condition exists because the total bending from end to end is zero when the ends are fixed. The second and third are true, because the total deflection is zero both vertically and horizontally, since the span is unvariable as well as the position of the tangents at the ends. These results are in accordance with Prop. V. Now by Prop. III these moments Me are the differences of the moments of a straight girder and of the arch itself; hence the polygon e is simply the polygon c in a new position and with a new pole distance. As moments are unchanged by such transformations, let us denote these moments by Mc. We have before seen that

$$\Sigma(Mc)=0, \text{ and } \Sigma(Mcx)=0$$

Subtract

$$\therefore \Sigma(Mc-Me)=0, \text{ and } \Sigma(Mc-Me)x=0$$

$$\therefore \Sigma(Md)=0 \text{ and } \Sigma(Mdx)=0$$

From this it is seen that the polygon d must have its closing line fulfill the same conditions as the polygon c. This is in accordance with Prop. IV.

$$\text{Again, } \Sigma(Mey) = \Sigma(Mc-Md)y=0$$

$$\therefore \Sigma(Mcy)=\Sigma(Mdy).$$

This last condition we shall use for determining the pole distance of the polygon e, which is one-third of the actual thrust of the arch measured on the scale of the weights w_1 w_2, etc. The physical significance of this condition may be stated according to Prop. V, thus: if the moments Md are applied to a uniform vertical girder bd at the points b, b_7'', b_6'', b_5'', etc., at the same height with b_8, d_7, etc., they will cause the same total deflection $xd = e.\Sigma(Mdy)$ as will the moments Mc when applied at the same points. Hence if Md are used as a species of loading, we can obtain the deflection by an equilibrium polygon. Suppose the load at d_7 is $d_7 k_7$, and that at d_6 is $d_6 k_6$, etc., then that at b_8 is $\frac{1}{2} b_8 k_8$. This approximation is sufficiently accurate for our purposes.

Now lay off on $l_8 l_8'$ as a load line $dm_8 = \frac{1}{2} b_8 k_8$, $m_8 m_7 = d_7 k_7$, $m_7 m_6 = d_6 k_6$, etc. The direction of these loads must be changed when they fall on the other side of the line k; *e.g.*, $m_5 m_4 = k_4 d_4$. If this process be continued through the entire arch m_5' (not drawn) will fall as far to the right of d as m_5 does to the left, and the last load will just reach to d again. This is a test of the correctness with which the position of the line $k_8 k_8'$ has been found. Now using any point as b for a pole, draw bm_8 to f_7, then draw $f_7 f_6 \parallel bm_7$, $f_6 f_5 \parallel bm_6$, etc. The curve bf is then the exaggerated shape of a vertical girder bd, fixed at b, under the action of that part of moments Md which are in the left half of the arch. The moments Md on the right may act on another equal girder, having the same initial position bd, and it will then be equally deflected to the right of bd. This is not drawn.

Again, suppose these vertical girders fixed at b are bent instead by the moments Mc. We do not know just how much these moments are, though we do know that they are proportional to the ordinates of the polygon c. Therefore make $dn_8 = \frac{1}{2} h_8 c_8$, $n_8 n_7 = h_7 c_7$, $n_7 n_6 = h_6 c_6$, etc. When all these loads are laid off, the last one $n_8' d = \frac{1}{2} h_8' c_8'$ must just return to d. This tests the accuracy of the work in determining the position of $h_8 h_8'$.

Now using b as a pole as before, construct the deflection curves bg and bg'. Since these two deflections, viz., 2 df and gg' ought to be the same, this fact informs us that each of the ordinates $h_1 c_1$, $h_2 c_2$, must be increased in the ratio of $\frac{1}{2} gg'$ to df, in order that when they are considered as loads, they may produce a total deflection equal to $2\,df$. To effect this, lay off $bj = df$ and $bi = \frac{1}{2} gg'$, and draw the horizontals through i and j. At any convenient distance draw the vertical $i_0 j_0$, and draw bi_0 and bj_0. These last two lines enable us to effect the required proportions for any ordinates on the left, and these or two lines of the same slope on the right to do the same thing on the right. *E. g.* lay off the ordinate $bi_8' = h_8' c_8'$, then the required new ordinate is bj_8'. Then lay off $k_8' e_8' = bj_8'$. In the same manner find $k e$ from $h b$, and $k_8 e_8$ from $h_8 c_8$. In the same manner can the other ordinates $k_1 e_1$, etc., be found; but this is not the best way to determine the rest of them, for we can now find the pole and pole distance of the polygon e.

As we have previously seen, the pole distance is decreased in the same ratio as the ordinates of the moment curve are increased, therefore prolong bi_0 to v_1, and draw a horizontal line through v_1 intersecting bj_0 at v_2 and the middle vertical at v_0; then is $v_2 v_0$ the pole distance decreased in the required ratio. Hence we move up the weight-line $w_1 w_8$ to the position $u_1 u_8$ vertically through v_2; and for convenience, lay off the weights $w_1' w_2'$ at $u_1' u_2'$, etc.

Furthermore, we know that the new closing-line is horizontal. To find the position of the pole o so that this shall occur, draw bv parallel to hh_8, and from v the horizontal vo. As is well known, v divides the total weight into the two segments, which are the vertical resistances of the abutments, and if the pole o is on the same horizontal with v, the closing line will be horizontal.

Now having determined the positions of the points e_8, e, e_8', starting from one of them, say e_8, draw $e_8 e_7 \parallel ou_8$, $e_7 e_6 \parallel ou_7$, etc.; then if the work be accurate, the polygon will pass through the other two points e and e_8'. The bending moments of the arch d or the arch a at a_1, a_2, etc., is the product of the pole distance $v_0 v_2 = v'o$ by the ordinates $d_1 e_1$, $d_2 e_2$, etc., respectively, and between these points a similar product gives the moment with sufficient accuracy. It would be useful for the sake of accuracy to multiply the ordinates of the arch by some number greater than 3.

As a final test of the accuracy of the work, let us see whether $\Sigma(Mey)$ is actually zero, as should be. At d_7, for example, $y = d_7 l_7$, and Me is proportional to $d_7 e_7$. Then $\overline{d_7 s_7}^2$ is proportional to Mey at that point if $e_7 s_7$ is the arc of a circle, of which $e_7 l_7$ is the diameter. Similarly find $d_7' s_7'$, etc. When e_4 for example falls above d_4, the circle must be described on the sum of $l_4 d_4$ and $d_4 e_4$ as a diameter, and $\overline{d_4 s_4}^2$ is proportional to a moment of different sign from that at d_7. We have distinguished the sign of the moments at the different points along the arch, by putting different signs before the letter s. It would have been slightly more accurate to have used only one-half the ordinates $b_8 e_8$ and $b_8' e_8'$, but as they nearly equal in this case and of opposite sign, we have introduced no appreciable error.

Now at any point s lay off $ss_7 = d_7 s_7$, and at right angles to it $s_7 s_8 = b_8 s_8$, then at right angles to the hypothenuse ss_8 make $s_8 s_5' = d_5' s_5'$, etc. Then the sum of the positive squares is ss_1', and similarly the sum of the negative squares is ss_1. If these are equal, then $\Sigma(Mey)$ vanishes as it should, and the construction is correctly made.

It would have been equally correct to suppose the two vertical girders fixed at d, and bent by the moments acting. We could have determined the required ratio equally well from this construction. Further, in proving the correctness of the construction by taking the algebraic sum of the squares, we could have reckoned the ordinates, y, from any other horizontal line as well as from $l_8 l_8'$.

To find the resultant stress in the different portions of the arch, we must prolong $v'o$ to o', say, (not drawn) so that the pole distance $v'o' = 3\,v'o$; then if we join o' and u_8, $o'u_8$ will be the resultant stress in the segment $b_8 a_7$; $o'u_7$ will be the stress in $a_7 a_6$, etc., measured in the same scale as the weights $w_1 w_2$, etc. This resultant stress is not directly along the neutral axis of the arch.

The vertical shearing stress is constructed in the same manner as for a girder, by drawing one horizontal through w_8 between the verticals 7 and 8, another through w_7 between 7 and 6, etc. (not drawn). Then the shear will be the vertical distance between *vo* and these horizontals through w_8, w_7, etc. It is seen that the shear will change sign on the vertical through b_1 with our present loading.

The actual position of the vertical through the center of gravity of the load may be found by prolonging the first and last sides of the polygon *c*. A weight $= \frac{1}{2} P = w_8\, w_9$ ought, however, first to be applied at b_2, and another $= \frac{1}{2} P = w_8'\, w_9'$ at b_8'. The shearing stress under a distributed load will actually change sign on the vertical so found. It will not pass far however from b_1.

The resultant stress is the resultant of the horizontal thrust and the vertical shearing stress, and it can be resolved into a tangential thrust along the arch and a normal shearing stress. This resolution will be effected in Fig. 3 of the next chapter.

As to the position of the moving load which will produce the maximum bending moments, we may say that the position chosen, in which the moving load covers one-half the span, gives in general nearly this case. It is possible, however, to increase one or two of the moments slightly by covering a little more than half the span with the moving load.

The loading which produces maximum moments will be treated more fully in subsequent chapters.

The maximum resultant stress and maximum vertical shear occur in general when the moving load covers the whole span. The construction in this case is much simplified, as the polygon *c* is then the same on the right of *b* as it now is on the left, and the center of gravity of the area is in the center vertical; so that the closing line $h_8\, h_8'$ is horizontal, and can be drawn with the same ease as $k_8\, k_8'$ was drawn. We shall not, even in this case, be under the necessity of drawing the curves *bg* and *bg'*, which would be both alike; for, as may be readily seen, the sum of the positive moments *Mc* on the left must be very approximately equal to the positive moments *Md* on the left, and the same thing is true for the negative moments at the left. The same two equalities hold also on the right. From this we at once obtain the ratio by which the ordinates of the polygon *c* must be altered to obtain those of the polygon *e*.

This last approximation also shows us that for a total uniform load, the four points of inflection when the bending moment is zero, lie two above and two below the closing line. It is frequently a sufficiently close approximation in the case when the moving load covers only part of the span to derive the ratio needed by supposing that the sum of all the ordinates, both right and left, above the closing line in the polygon *c* must be increased, so that it shall equal the corresponding sum in the polygon *d*. If the sums taken below the closing lines give a slightly different result, take the mean value.

Thus the single construction we have given in Fig. 2, and one other much simpler than this, which can be obtained by adding a few lines to Fig. 2, give a pretty complete determination of the maximum stresses on the assumptions made at the commencement of the article.

One of these assumptions, viz., that of constant cross section (*i. e.* I=constant), deserves a single remark. In the St. Louis Arch I was increased one-half at each end for a distance of one-twelfth of the span. This very considerable change in the value of I slightly reduced the maximum moments computed for a constant cross section. From other elaborate calculations, particularly those of Heppel,[*] on the Britannia Tubular Bridge, it appears that the variation in the moments caused by the changes in cross section, which will adapt the rib to the stresses it must sustain, are relatively small, and in ordinary cases are less than five per cent. of the total stress. The same considerations are not applicable near the free ends of a continuous girder, where I may theoretically vanish. In the case before us, where the principal part of the stress arises not from the bending moments, but from the compression along the arch, the effect of the variation of I is very inconsiderable indeed.

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[*] *Philosophical Magazine*, Vol. 40, 1870.

CHAPTER III.

ARCH RIB WITH FIXED ENDS AND HINGE JOINT AT THE CROWN.

LET the curve a of Fig. 3 represent the proportions of the arch we shall use to illustrate the method to be applied to arches of this character. The arch a is segmental in shape, and has a rise of one-fifth of the span. It is unnecessary to assume the particular dimensions in feet, as the above ratio is sufficient to determine the shape of the arch.

The arch is supposed to be fixed in the abutments, in such a manner that the position of a line drawn tangent to the curve a at either abutment is not changed in direction by any deflection which the arch may undergo. At the crown, however, is a joint, which is perfectly free to turn, and which will, then, not allow the propagation of any bending moment from one side to the other. In order that we may effect the construction more accurately, let us multiply the ordinates of the curve a by some convenient number, say 2, though a still larger multiplier would conduce to greater accuracy. We thus obtain the polygon d.

Having divided the span b into twelve equal parts b_1 b_2, etc, (a larger number of parts would be better for the discussion of an actual case), we lay off below the horizontal line b on the end verticals, lengths which express on some assumed scale the weights which may be supposed to be concentrated at the points of division of the arch. If $a\,l$ is the depth of the loading on the left and $al' = \frac{1}{2}al$ that on the right, then $b_6 w_1 + b_6' w_6' =$ the weight concentrated at a; $w_1 w_2 =$ the weight at a_1; $w_1'\ w_2' =$ the weight at a_1', etc. Using b as a pole, draw the equilibrium polygon c, whose extremities c_6 and c_6' bisect $w_3\ w_4$ and $w_3'\ w_4'$ respectively.

Now to find the closing line of this equilibrium polygon so that its ordinates shall be proportional to the bending moments of a straight girder of the same span, and of a uniform moment of inertia I, which is built in horizontally at the ends and has a hinge joint at its center; we notice in the first place that the bending moment at the hinge is zero, and hence the ordinate of the equilibrium polygon at this point vanishes. The closing line then passes through b the point in question. Furthermore it is evident that if we consider the parts of the girder at the right and left of the center as two separate girders whose ends are joined at the center, these ends have each the same deflection, by reason of this connection.

This is expressed by means of our equations by saying that $\Sigma(Mx)$ when the summation is extended from one end to the center is equal to $\Sigma(Mx)$ when the summation is extended from the other end to the center, for these are then proportional to the respective deflections of the center. We may then write it thus :

$$\Sigma_{b_6}^{b}\,(Mx) = \Sigma_{b_6'}^{b}(Mx)$$

The equation has this meaning, viz : that the center of gravity of the right and left moment areas taken together is in the center vertical : for, taking each moment M as a weight, x is its arm, and Mx its moment about the center.

In order to find in what direction to draw the closing line through b so that it shall cause the moment areas together to have their center of gravity in the center vertical through b, let us draw a second equilibrium polygon using the moment areas as a species of loading.

The area on the left included between any assumed closing line as bb_6 (or bh_6) and the polygon bc_6 may be considered to consist of a positive triangular area bc_6b_6 (or bc_6h_6) and a negative parabolic area $bc_3c_6c_0$; and similarly on the right a positive area $bc_6'b_6'$ (or $bc_6'h_6'$) and a negative area $bc_3'c_6'c_0'$.

At any convenient equal distances from the center as at p and p', lay off these loads to some convenient scale. It is, perhaps, most convenient to reduce the moment areas to equivalent triangles having each a base equal to half the span: then take the altitudes of the triangles as the loads. This we have done, so that $pp_1 = \frac{4}{3}c_0c_3$, and $p'p_1' = \frac{4}{3}c_0'c_3'$. Now assume, for the instant, that closing line is b_6b_6', which of course is incorrect, and make $p_1p_2 = b_6c_6$ and $p_1'p_2' = b_6'c_6'$, then these are the loads due to the positive triangular areas at the left and right respectively, while pp_1 and $p'p_1'$ are the negative parabolic loads.

Take o' as the pole of these loads, then pp' may be taken for the first side of the second equilibrium polygon. Draw $pq \parallel o'p_1$ and $p'q' \parallel o'p_1'$, and then from q

and q' draw parallels to $o'p_2'$ respectively. These last sides intersect at q_2. The vertical through q_2 then contains the center of gravity of the moment areas when $b_6\ b_6'$ is assumed as the closing line.

A few trials will enable us to find the position of the closing line which causes the center of gravity to fall on the center vertical. We are able to conduct these trials so as to lead at once to the required closing line as follows. Since, evidently, $b_6c_6+b_6'c_6'=h_6c_6+h_6'c_6'$, it is seen that the sum of the positive loads is constant. Therefore make $p_2p_3=p_2'p_3'$ and use p_1p_3 and $p_1'p_3'$ as the positive loads, in the same manner as we used p_1p_2 and $p'_1p'_2$ previously.

This will be equivalent to assuming a new position of the closing line. The only change in the second equilibrium polygon will be in the position of the last two sides. These must now be drawn parallel to $o'p_3$ and $o'p_3'$ respectively; and they intersect at q_3. The vertical through q_3 contains the center of gravity for this assumed closing line. Another trial gives us q_4.

Now if the direction of the closing line had changed gradually, then the intersection of the last sides of the second equilibrium polygon would have described a curve through q_2, q_3 and q_4. If one of these points, as q_3, is near the center vertical, then the arc of a circle q_2q_3 q_4, will intersect it at q_5 indefinitely near to the point where the true locus of the points of intersection would intersect the center vertical.

Let us assume that q_5 is then determined with sufficient exactness by the circular arc $q_2q_3q_4$, and draw qq_5 and $q'q_5$ as the last two sides of the second equilibrium polygon. Now draw $o'p_5 \parallel qq_5$ and $o'p_5' \parallel q'q_5$, then $p_1p_5=c_6h_6$ and $p_1'p_5'=c_6'h_6'$ are the required positive loads, and h_6bh_6' is the position of the closing line such that the center of gravity of the moment areas is in the center vertical.

It is evident that the closing line of the polygon d considered as itself an equilibrium polygon is the horizontal line through d, for that will cause the center of gravity of the moment areas on the left and right, between it and the polygon d, to fall on the center vertical.

The next step in the construction is to apply Prop. IV, for the determination of the bending moments.

That Prop. IV is true for an arch of this kind is evident; for, the loading causes bending moments proportional to the ordinates h_2c_2, h_3c_3, etc., while the arch itself is fitted to neutralize, in virtue of its shape, moments which are proportional to k_2d_2, k_3d_3, etc. The differences of the moments represented by these ordinates are what actually produce bending in the arch.

Now the ordinates of the type hc are not drawn to the same scale as those of the type kd, for each was assumed regardless of the other. In order that we may find the ratio in which the ordinates hc must be changed to lay them off on the same scale as kd it is necessary to use another equation of condition imposed by the nature of the joint and supports, viz:

$$\Sigma_{b_6}^{a}\,(M_a-M_c)y=\Sigma_{b_6'}^{a}(M_a-M_c)y$$

or $$\Sigma_{b_6}^{d}\,(M_d-M_c)y=\Sigma_{b_6'}^{d}(M_d-M_c)y$$

The left hand side of the equation is the horizontal displacement (*i.e.*, the total deflection) of the extremity a of the left half of the arch, due to the actual bending moments (M_d-M_c) acting upon it: and the right hand side is the horizontal displacement of a the extremity of the right half of the arch due to the moments actually bending it. These are equal because connected by the joint.

The construction of the deflection curves due to these moments will enable us to find the desired ratio.

The ordinates kd and hc are rather longer than can be used conveniently, to represent the intensity of the moments concentrated at d_1,d_2, etc, and c_1,c_2, etc.: so we will use the halves of these quantities instead. Therefore lay off $dm_6=\frac{1}{4}k_6b_6$, $m_6m_5=\frac{1}{2}k_5\,d_5$, $m_5m_4=\frac{1}{2}k_4d_4$, etc., and also $dn_6=\frac{1}{4}h_6c_6$, $n_6n_5=\frac{1}{2}h_5c_5$, etc.

We use only one-quarter of each end ordinate because the moment area supposed to be concentrated at each end has only one half the width of the moment areas concentrated at the remaining points of division.

Using b as a pole we find the deflection curve fb due to the moment M_a or M_d and the deflection curve gb due to the moments M_c on the left. On the right we should find a deflection $df'=df$ not drawn, and similarly a deflection dg' not equal to dg.

Now the equation we are using requires that the ordinates hc shall be elongated so that when used as weights the deflections shall be identical: *i.e.*, we must have $df=\frac{1}{2}gg'$. To effect the elongation, lay off $aj=df$ and $ai=\frac{1}{2}gg'$; and at any convenient distance on the horizontals ii_0 and jj_0 draw the vertical i_0j_0; then the lines ai_0 and aj_0 will effect the required elongation. For example, lay off $ai_6=h_6c_6$, from which we obtain $aj_6=k_6e_6$ for the left end ordinate, and similarly $aj_6'=k_6'e_6'$.

The pole distance tt_1, of the original polygon c must be shortened in the same ratio in which the ordinates are elongated. Hence the new pole distance of the polygon e is tt_2.

Since k_6k_6' is the closing line of the polygon e, and is horizontal, the pole of e is o, on the horizontal through h_6; for, h_6w_6 is the part of the applied weight sustained by the left support.

Now if the weight line be moved up to t_2 so that the applied weights are u_1u_1' at the center, etc., and o is the pole, the polygon e may be described starting from d, and it will finally cut off the end ordinates k_6e_6 and $k_6'e_6'$ before obtained. Then will the ordinates of the type de be proportional to the moments actually bending the arch, and the moments will be equal to the products of de by tt_2, in which de is measured on the scale of distance, and tt_2 on the scale adopted for the weights w_1w_2, etc.

The accuracy of the construction is finally tested by taking $\Sigma(ds)^2=0$, an equation deduced from $\Sigma''(M_d-M_c)y=0$, as explained in the previous article upon the St. Louis Arch. It is unnecessary to explain the details of this construction since as appears from Fig. 3 it is in all respects like that in Fig. 2.

Now let us find the intensity of the tangential compression along the arch and of the shearing normal to the arch. Since the pole distance tt_2 refers to the difference of ordinates between the polygons d and e, whose ordinates are double the actual ordinates, if we wish now to return to the actual arch a whose ordinates are halves of the ordinates of d, we must take a pole distance $tt_3=2tt_2$ and move the weight line so that it is the vertical through t_3. Then tt_3 is the actual horizontal thrust of this arch due to the weights; and ov_6 is the resultant stress in the segment a_6b_6 of the arch, which may be resolved into two components or_6 and v_6r_6 respectively parallel and perpendicular to a_6b_6.

Then are or_6 and v_6r_6 respectively, the thrust directly along, and the shear directly across the segment a_6b_6 of the arch. Similarly or_5 and v_5r_5 represent the thrust along, and the shear across the segment a_5a_4, and so on for other segments. These quantities are all measured in the same scale as that of the applied weights.

The shear changes sign twice, as will be seen from inspection of the directions in which the quantities of the type vr are drawn. The shear is zero wherever the curves d and e are parallel to each other. Thus the shear is nearly zero at b_6, at a_2 and at some point between a_2' and a_3'.

The maxima and minima shearing stresses are to be found where the inclination between the tangents to the curves d and e are greatest.

The statements made in the previous article, respecting the position of the moving load which causes maximum bending moments, are applicable to this kind of arch also.

The maximum normal shearing stress will occur for the parts of the arch near the center, when the moving load is near its present position, covering one half of the arch. But the maximum normal shearing stress near the ends, may occur when the arch is entirely covered by the moving load, or when it may occur when the moving load is near its present position, it being dependent upon the rise of the arch, and the ratio between the moving and permanent load.

The maximum tangential compressions occur when the moving load covers the entire arch. The stresses obtained by the foregoing constructions, go upon the supposition that the arch has a constant cross-section, so that its moment of inertia does not vary, and no account is taken of the stresses caused by any changes of the length of the arch rib, due to variations of temperature or other causes. These latter stresses we shall now investigate for both of the kinds of arches which have been treated.

CHAPTER IV.

TEMPERATURE STRAINS.

It is convenient to classify all strains and stresses arising from a variation in the length of the arch, under the head of temperature, as such stresses could evidently have been brought about by suitable variations of temperature.

The stresses of this kind which are of sufficient magnitude to be worthy of consideration, besides temperature stresses are of two kinds, viz. the elastic shortening of the arch under the compression to which it is subjected, and the yielding of the abutments, under the horizontal thrust applied to them by the arch. This latter may be elastic or otherwise. It was, I believe, neglected in the computation of the St. Louis Arch, and no doubt with sufficient reason, as the other stresses of this kind were estimated with a sufficient margin to cover this also. Anything which makes the true span of the arch differ from its actual span causes strains of this character. By true span is meant the span which the arch would have if laid flat on its side on a plane surface in such a position that there are no bending moments at any point of it, while the actual span is the distance between the piers when the arch is in position. If the arch be built in position, but joined at the wrong temperature the true and actual spans do not agree and excessive temperature strains are caused.

Taking the coefficient of expansion of steel as ordinarily given, a change of ±80°F. from the mean temperature would cause the St. Louis Arch to be fitted to a span of about $3\frac{1}{4}$ inches, greater or less than at the mean.

The problem we wish to solve then is very approximately this: What horizontal thrust must be applied to increase or decrease the span of this arch by $3\frac{1}{4}$ inches, and what other stresses are induced by this thrust. In Fig. 4 the half span is represented on the same scale as in Fig. 2. The only forces applied to the half arch are an unknown horizontal thrust H at b_8 and an equal opposite thrust H at a. The arch is in the same condition as it would be if Fig. 4 represented half of a gothic arch of a span = $2ab$, of which a was one abutment, and b_8 was the new crown at which a weight of $2H$ was applied. The gothic arch would be continuous at the crown, but the abutment a would be mounted on rollers, so that although the direction of a tangent at a could not be changed, nevertheless the abutment could afford no resistance to keep the ends from moving apart, *i.e.* there is no thrust in the direction of ab, any more than there is along an ordinary straight girder.

In order to facilitate the accurate construction, let us multiply the ordinates of a by 3 and use the polygon d instead. Now the real equilibrium polygon of the applied forces H, is the straight line kk_8. By real equilibrium polygon is meant, that one which has for its pole distance, the actual thrust of the arch. As we are now considering this arch, H is the applied force, and the thrust spoken of is at right angles to H. We have just shown this thrust to be zero. We have then to construct an equilibrium polygon for the applied force H with a pole distance of zero. The polygon is infinitely deep in the direction of H, and hence is a line parallel to H. This fixes its direction.

Its position is fixed from the consideration that the total bending is zero, (because the direction of the tangents at the extremities a and b_8 are unchanged), which is expressed by the equation

$$\Sigma(M_d)=0.$$

This gives us the same closing line through k which we found in Fig. 2, and the ordinates of the type kd, are proportional to the moments caused by the horizontal thrust H.

Now lay off $dm_8=\frac{1}{2}k_8b_8$, $m_8m_7=k_7d_7$, etc., as in Fig. 2.

The problem of finally determining H, will be solved in two steps:

1°. We shall find the actual values of the moments to which the ordinates kd are proportional;

2°. We shall find H by dividing either of these moments by its arm.

By considering the equation

$$D_yEI=\Sigma(My)$$

given in Chapter I, in which D_y is the horizontal displacement, it is seen that if the actual moments are used for weights, and EI for the pole distance, we shall obtain, as the second equilibrium polygon, a deflection curve whose ordinates are the actual deflections due to the moments. By actual moments, actual deflections, etc, is meant, that all of the quantities in the equation are laid off to the scale of distance, say *one* n^{th} of the actual size.

Now let the equation be written

$$nD_y \cdot \frac{1}{n} EI = \Sigma(My).$$

From which it is seen that if the ordinates be multiplied by n, so that on the paper they are of the same size as in the arch, we must use *one* n^{th} of the former pole distance, all else remaining unchanged.

Now for the St. Louis Arch, $EI=$ 39680000 foot tons. Let us take 100 tons to the inch, as the scale of force: and since $bd=3$ inches, the scale of distance n is found from the proportion

$$3 \text{ in.} :: 51.8 \text{ ft.} :: 1 : n = 210 \text{ nearly,}$$
$$\text{and } EI \div 100\, n^2 = 9 \text{ in. nearly,}$$

which is the pole distance necessary to use with the actual deflection $\frac{1}{2}$ of $3\frac{1}{4}$ in.$=$ $1\frac{5}{8}$in., in order that the moments may be measured to scale. As it is inconvenient to use so large a distance as 9 in. on our paper, let us take $\frac{3}{8}$ of 9 in. $=3\frac{1}{3}$ in. nearly $=dz$ for the pole distance, and $\frac{8}{3}$ of $1\frac{5}{8}$ in. $=4\frac{1}{3}$ in. $=dy$, for the deflection.

Now with z as a pole and the weights dm_8, m_8m_7, etc, draw the deflection curve bf, having the deflection $=df$. The moments M_d must be increased in such a ratio that the deflection will be increased from df to dy. Therefore draw the straight lines bf and by, which will enable us to effect the increase in the required ratio. For example, the moment $dm_1=bi$ is increased to bj, and $dm_8=bj$ is increased to bj_8. Now measuring bj in inches and multiplying by 210 and by 100, we have found that 21000 $bj=1809$ foot tons$=$the moment at d or a.

And again, 21000 $bj_8=3747$ foot tons $=$the moment at b_6.

By measurement 210 $dk=17$ ft. and 210 $bk=34.8$ ft.

$$\therefore H = 1809 \div 17 = 106 \text{ tons, } +$$
$$\text{or } H=3747 \div 34.8 = 108 \text{ tons } -.$$

These results should be identical, and the difference between them of less than 2 per cent. is due to the error occasioned by using the polygon d instead of the curve of the ellipse, and to small errors in measurement. With a larger figure and the subdivision of the span into a greater number of parts this error could be reduced. The value of H found for the St. Louis Arch by computation was 104 tons, but that was not on the supposition of a uniform moment of inertia I, and should be less than the value we have obtained.

Now this horizontal thrust H due to temperature and to any other thrusts of like nature as compression, etc, is of the nature of a correction to the thrust due to the applied weights. Thus in Fig. 2 we found $3ov'$ to be the thrust due to the applied weights, and on applying the correction we must use the two thrusts $3ov'+H$ and $3ov'-H$ as pole distances to obtain equilibrium polygons whose ordinates reckoned from the arch a will, when multiplied by its pole distance, give the true bending moments. The tangential and normal stresses can then be determined by resolution, precisely as in Fig. 3.

If it, however, appears desirable to compute separately the strains due to H, this may be more readily done than in combination with the stresses already obtained. We have already seen sufficiently how the bending moments due to H are found. In fact the moments are such as would be produced by applying H at the point where the horizontal through k cuts the polygon d, for this is the point of no moment, and may be considered for the instant as a free end of each segment, to each of which H is applied causing the moments due to its arm and intensity.

To find the tangential stress and shear, lay off in Fig. 4 $av=H$ and on it as a diameter describe a semicircle, and draw $ar_8 \parallel a_7a_8$, $ar_7 \parallel a_6a_7$ etc.; then will ar_8 be the component of H along a_7a_8, and vr_8 be the component of H directly across the same segment. In a similar manner the quantities of which ar on the type are the tangential stresses and the quantities vr are the shearing stresses caused by H.

The scale used for this last construction is about fifty tons to the inch.

Now H is positive or negative according as the temperature is increased above or diminished below the mean, and these tangential and normal components, of course, change sign with H.

It should also be noticed in this connection that thrusts and bending moments, which are numerically equal but of opposite sign, are induced by equal contractions and expansions.

The stresses due to variation of temperature in the arch of Fig. 3, having a center joint, are constructed in Fig. 5.

It is evident from reasoning similar to that employed for the case just discussed, that the closing line dk_6 of the polygon d is the equilibrium polygon of the thrust H induced by variation of temperature. Suppose we have changed the equation of deflections to the form,

$$mD_y \cdot \frac{EI}{mn^2} = \Sigma\left(\frac{M}{n} \cdot \frac{y}{n}\right),$$

in which, if $mD_y = dy$ and $EI \div mn^2 = dz$, then the moments M and the ordinates y will be laid off on the scale of 1 to n. This is equivalent to doing what was done in the previous case, where m was equal to $\frac{8}{3}$. The remainder of the process is that previously employed.

It should be noticed that we have in Figs. 4 and 5, incidentally discussed two new forms of arches, viz: in Fig. 4 that of an arch having its ends fixed in direction, but not in position; *i.e.*, its ends may slide but not turn, and in Fig. 5, that of an arch sliding freely and turning freely at the ends. The first of these arches has the same bending moments as a straight girder, fixed in direction at the ends, and the second of them has the same bending moments as a simple girder supported at its ends.

Errata.—The measurements of Fig. 4 given on page 24 do not agree with the scale on which the drawing is engraved. The following equations and quantities agree with the dimensions of Fig. 4, and are to be substituted instead of those given on page 24.

Let the scale of force be 100 tons to the inch, and since $bd = 4\frac{1}{2}$ inches, $4\frac{1}{2}$ in. : 51.8 ft. :: 1 : $n = 140$ nearly, and $EI \div 100n^2 = 20$ in. nearly, which is the pole distance to use with the actual deflection of the half span $= 1\frac{5}{8}$ in.

Now take one fourth of this pole distance $= 5$ in. $= dz$, and four times the deflection $= 6\frac{1}{2}$ in. $= dy$, as being more convenient to use; the moments, which are the products of the deflections by the pole distance, will be unchanged by this process.

Now increase the ordinates in such a ratio that the deflection will be increased from df to dy. For example, the moment $dm_1 = bi$ is increased to bj, and $dm_8 = bi_8$ is increased to bj_8. Now by measuring bj in inches and multiplying by 140 and by 100 we have found $14000\ bj = 1809$ foot tons = the moment at a or d. And again, $14000\ bj_8 = 3747$ foot tons = the moment at b_8.

By measurement, $140\ dk = 17$ ft.

and $140\ bk = 34.8$ ft.

$\therefore\ H = 1809 \div 17 = 106$ tons $+$,

or $H = 3747 \div 34.8 = 108$ tons $-$.

Near the bottom of the second column of page 24, instead of ar_8, ar_7, vr_8, ar, vr, read av_8. av_7, vv_8, av, vv.

The scale used in the last construction in Fig. 4, is about $33\frac{1}{3}$ tons to the inch.

UNSYMMETRICAL ARCHES.

The constructions which have been given have been simplified somewhat by the symmetry of the right and left hand halves of the arch, but the methods which have been used are equally applicable if such symmetry does not exist, as it does not, if, for example, the abutments are of different heights.

In particular, for the unsymmetrical arch, its closing line is not in general horizontal, and must be found precisely as that for the equilibrium polygon due to the applied weights.

If, in Fig. 3, the hinge joint is not situated at the center, the arch is unsymmetrical, and the determination of the closing line due to the applied weights, is not quite so simple as in Fig. 3. It will be necessary to draw the trial lines through the joint by which the curve of errors q is found.

CHAPTER V.

ARCH RIB WITH END JOINTS.

Let the curve a of the arch to be treated have a span of six times the rise, as represented in Fig. 6, and having divided the span into twelve equal parts, make the ordinates of the type bd twice the ordinates ab.

Let a uniform load having a depth xy cover the two-thirds of the span at the left, and a uniform load having a depth

$xy' = \frac{1}{2} xy$ cover the one-third of the span at the right. Assume any pole distance, as of one-third of the span, and lay off $b_4 w_1 = xy$ = one-half of the load supposed to be concentrated at the center; $w_1 w_2 = 2\,xy$ = the load concentrated above b_1, etc. Similarly at the left make $b_4' w_1' = xy$ = one-half the load above b; $w_1' w_2' = 2\,xy$ = the load above b_1'; $w_2' w_3' = xy + xy' = \frac{3}{2} xy$ = the load above b_2'; $w_3' w_4' = xy$ = the load above b_3', etc.

From this force polygon draw the equilibrium polygon c, just as in Figs. 2 and 3.

Now the closing line of the equilibrium polygon for a straight girder with ends free to turn, must evidently pass so that the end moments vanish. Hence $c_6 c_6'$ is the closing line of the polygon c, and $b_6 b_6'$ is the closing line of the polygon d, drawn according to the same law. The remaining condition by which to determine the bending moments is:

$$\Sigma\,(M_d - M_c)y = 0 \quad \therefore \quad \Sigma(M_d y) = \Sigma(M_c y)$$

which is the equation expressing the condition that the span is invariable, the summation being extended from end to end of the arch.

This summation is effected first as in Figs. 2 and 3, by laying off as loads quantities proportional to the applied moments concentrated at the points of division of the arch, and thus finding the second equilibrium polygon, or deflection polygon of two upright girders, bent by these moments.

Let us take one-fourth of each of the ordinates bd for these loads, *i.e.* $bm = \frac{1}{4}$ of $\frac{1}{2}\,bd$; $mm_1 = \frac{1}{4}\,b_1 d_1$, etc.: also bn, nn_1, etc., equal to similar fractions of the ordinates of the curve c. Using d as the pole for this load, we obtain the total deflection bf_6 on the left, and the same on the right (not drawn) due to the bending moments M_d.

Similarly $g_6 g_6'$ is the total deflection right and left due to the moments M_c.

Now the equation of condition requires that $\frac{1}{2}\,g_6 g_6' = bf_6$. That this may occur, the ordinates of the polygon c must be elongated in the ratio of these deflections. To effect this, make $ai = \frac{1}{2}\,g_6 g_6'$ and $aj = bf_6$, and on the horizontals through i and j at a convenient distance draw the vertical $i_0 j_0$; then the lines ai_0 and aj_0 will effect the required elongation, as previously explained. To obtain the center ordinate be, for example, make $ai' = bh \;\therefore\; aj' = be$. To find the new pole o, draw bv parallel to $c_6 c_6'$ and vo horizontal, as before explained.

If ai_0 cuts the load line at t_1 and the horizontal through t_1 cuts aj_0 at t_2, then the vertical through t_2 is the new position of the load line and tt_2 is the new horizontal thrust.

Now using o as the pole of the load line $u_1 u_1'$ etc., through t_2 draw the equilibrium polygon starting from e. It must pass through b_6 and b_6', which tests the accuracy of the construction.

The construction may now be completed just as in Fig. 3, by doubling the pole distance, and finding the tangential thrust along the arch and the normal shear directly across the arch in the segments into which it is divided. The maximum thrust and tangential stress is obtained when the line load covers the entire span.

To compute the effect of changes of temperature and other causes of like nature in producing thrust, shear, bending moment etc., let us put the equation of deflections in the following form:

$$mD_y \cdot \frac{EI}{mn^2 n'} = \Sigma\left(\frac{M}{nn'} \cdot \frac{y}{n}\right) \quad . \quad \text{(D)}$$

This equation may perhaps put in more intelligible form the processes used in Figs. 4 and 5, and is the equation which should be used as the basis for the discussion of temperature strains in the arch. In equation (D) n is the number by which the rise of the arch must be divided to reduce it to bd, *i.e.*, it is the scale of the vertical ordinates of the type bd, in Fig. 6, so that if bd was on the same scale as the arch itself, n would be unity. Again, n' is the scale of force, *i.e.*, the number of tons to the inch; and m is a number introduced for convenience so that any assumed pole distance p may be used for the pole distance of the second equilibrium polygon. In Fig. 6, $p = bd$.

We find m from the equation,

$$p = \frac{EI}{mn^2 n'} \quad \therefore \quad m = \frac{EI}{pn^2 n'},$$

from which m may be computed, for EI is a certain known number of foot tons when the cross-section of the rib is given, p is a number of inches assumed in the drawing, n and n' are also assumed. Now D_y is the number of inches by which the span is increased or decreased by the change of temperature, and mD_y is at once laid off on the drawing.

The quantities in equation (D) are so related to each other, that the left-hand member is the product of the pole distance and ordinate of the second equilibrium polygon, while the right-hand member is the bending moment produced by the loading $M \div nn'$, which loading is proportional to M. The curve f was constructed with this loading, and only needs to have its loads and ordinates elongated in the ratio of bf_6 to $\frac{1}{2}\, mD_y$ to determine the values of $M \div nn'$ at the various points of division of the arch. One-half of each quantity is used, because we need to use but one-half the arch in this computation. Two lines drawn, as in Figs. 4 and 5, effect the required elongation.

The foregoing discussion is on the implied assumption that the horizontal thrust caused by variation of temperature is applied in the closing line bb_6 of the arch, which is so evident from previous discussions as to require no proof here.

The quantity determined by the foregoing process is $M \div nn' = q$ say, a certain number of inches. Then $M = nn'q$, and $H = M \div y = n'q \div \frac{y}{n}$, in which $\frac{y}{n}$ is the length of the ordinate in inches on the drawing at the point at which M is applied.

The determination of the shearing and tangential stress induced by H is found by using H as the diameter of a circle, in which are inscribed triangles, whose sides are respectively parallel and perpendicular to the segments of the arch, precisely as was done in Figs. 4 and 5.

The whole discussion of the arch with end joints may be applied to an unsymmetrical arch with end joints. In that case, it would be necessary to draw a curve f' at the right as well as f at the left, and the two would be unlike, as g and g' are. This, however, would afford no difficulty either in determining the stresses due to the loads, or to the variations of temperature.

When the live load extends over two-thirds of the span, as in the Fig., the maximum bending moment is nearly in the middle of that live load, and is very approximately the largest which can be induced by a live load of this intensity, while the greatest moment of opposite sign is found near the middle of the unloaded third of the span.

If the curve of the arch were a parabola instead of the segment of a circle, these statements would be exact and not approximate, as may be proved analytically. This matter will be further treated hereafter.

CHAPTER VI.

ARCH RIB WITH THREE JOINTS.

Let the joints be at the center and ends of the arch, as seen in Fig. 7. Let the loading and shape of the arch be the same as that used in Fig. 6. Now since the bending moment must vanish at each of the joints, the true equilibrium curve must pass through each of the joints; *i. e.*, every ordinate of the polygon c must be elongated in the ratio of db to bh. To effect this, make $di = bh$, and at a convenient distance on the horizontals through b and i draw the vertical $i_0 b_4$. Then the ratio lines di_0 and db_4 will enable us to elongate as required, or to find the new pole distance ti, diminished in the same ratio, by drawing the horizontal ti through i_0. The new pole o is found in the same manner as in Fig. 6.

Now with the new pole o and the new load line through t, we can draw the polygon e starting at d. It must then pass through b_6 and b_6' which tests the accuracy of the construction.

The maximum thrust, and tangential stress is attained when the live load covers the entire span.

Variations in length due to changes of temperature induce no bending moments in this arch, but there may be slight alteration in the thrust, etc., produced by the slight rising or falling of the crown due to the elongation or shortening of the arch. This is so small a displacement that it is of no importance to compute the stresses due to it. We have for the same reason, in the previous and subsequent constructions, omitted to compute the stresses arising from the displacement which the arch undergoes at various points by reason of its being bent. It would be quite possible to give a complete investigation of these stresses by analogous methods.

The construction above given is applicable to any arch with three joints. The arch need not be symmetrical, and the three joints can be situated at any points of the arch as well as at the points chosen above.

CHAPTER VII.

THE ARCH RIB WITH ONE END JOINT.

Let the arch be represented by Fig. 8, in which the load, etc., is the same as in Fig. 6.

The closing line must pass through the joint, for at this joint the bending moment vanishes.

A second condition which must be fulfilled is, that the total deflection below the tangent at the fixed end of a straight girder having one end joint vanishes, for the position of the joint is fixed. This is expressed by the equation

$$\Sigma(Mx)=0,$$

in which the summation is extended from end to end.

This condition will enable us to draw the closing line of the polygon *c*, and also that of *d*. The problem may be thus stated:—In what direction shall a closing line such as c_6h' be drawn from c_6 so that the moment of the negative triangular area $c_6c_6'h'$ about c_6 shall be equal to the moment of the positive parabolic area c_6bc_6'

To solve this problem, first find the center of gravity of the parabolic area by taking it in parts. The parabolic area $c_6\ b\ c_2'$ is a segment of a single parabola whose area is $\frac{2}{3}b_6b_2' \times c_0c_2 = \frac{1}{2}h_1 \times b_6b_6'$, when h_1 = the height of an equivalent triangle having the span for its base $\therefore\ h_1 = \frac{8}{9}c_0c_2$.

Lay off $l_6b_6 = c_0c_2$, and draw l_6b_3' $\therefore$ $b_5l_5 = h_1$. Lay off $c_2'p_1 = h_1$ as proportional to the weight of the parabolic area. Again, $c_2'p$ is proportional to the weight of the triangle $c_6c_2'c_6'$. The parabolic area $c_2'c_6' = \frac{2}{3}c_0'c_4' \times b_2'b_6' = \frac{1}{2}h_2 \times b_6b_6'$, as before, $\therefore$ $h_2 = \frac{4}{9}c_0'c_4'$, which may be found as h_1 was before.

Let $h_2 == pp_2$, then on taking any pole, as c_2, of this weight line, we draw $qq_1 \parallel c_2c_2'$, since the left parabolic area has its center of gravity in the vertical through q_1, and the triangular area in that through q, we draw $qq_1' \parallel c_2p$, to the vertical through q_1', which contains the center of gravity of the right parabolic area. The position of q midway between the verticals containing b and b_2 is slightly to the right of its true position, as it should be at one-third of the distance from the vertical through b to that through b_2. This does not affect the nature of the process however.

Then $q_1q_2 \parallel c_2p_1$ and $q_1'q_2 \parallel c_2p_2$ give q_2 in the vertical through the center of gravity of the total positive area. The negative area, since it is triangular, has its center of gravity in the vertical through c_2'.

Now if the total positive bending moment be considered to be concentrated at its center of gravity and to act on a straight girder it will assume the shape rq_2r_1 of this second equilibrium polygon, and if a negative moment must be applied such that the deflection vanish, the remainder of the girder must be r_1r_2, a prolongation of rr_1. Now draw $c_2p_3 \parallel rr_1$, and we have $p_2p_3 = c_6'h'$ the height of the triangle of negative area. Hence c_6h' is the closing line, fulfilling the required conditions.

Again, to draw the closing line b_6k' according to the same law, we know that the center of gravity of the polygonal area *d* is in the center vertical. To find the height p_1p', of an equivalent triangle having a base equal to the span, we may obtain an approximate result, as in Fig. 2, by taking one twelfth of the sum of the ordinates of the type *bd*, but it is much better to obtain an exact result by applying Simpson's rule which is simplified by the vanishing of the end ordinates. The rule is found to reduce in this case to the following:—The required height is one eighteenth of the sum of the ordinates with even subscripts plus one ninth of the sum of the rest.

Now this positive moment concentrated in the center vertical and a negative moment such as to cause no total deflection in a straight girder, will give as a second equilibrium polygon $rq_2'r_1'r_2'$; and if $c_2p_3' \parallel rr_1'$, then $p'p_3' = b_6'k'$ is the height of the triangular negative area, and the closing line is b_6k'.

Now the remaining condition is that the span is invariable, which is expressed by the equation

$$\Sigma(M_d - M_c)y = 0, \text{ or } \Sigma(M_dy) = \Sigma(M_cy).$$

Let us construct the deflection curve due to the moments M_d in a manner similar to that employed in Fig. 2. We lay off quantities dm_5, m_5m_4, etc., equal to one-fourth of the corresponding ordinates of the curve d, and dn_5, n_5n_4, etc., one-fourth of the ordinates of the curve c. We use one-fourth or any other fraction or multiple of both which may be convenient. By using b for a pole we obtain the deflection curves f and f' for the moments proportional to M_a, and the curves g and g' for those proportional to M_c.

Now, Prop. IV. requires that the ordinates of the polygon c should be increased so that gg' shall become equal to ff'. Make $di=gg'$ and $dj=ff'$ and draw as before the ratio lines di_0 and dj_0, then the vertical through t_1 is the new position of the load line.

Find the new length of bh which is ke, and with the new pole o, draw the polygon e starting at e. It must pass through b_6. The new pole o is found thus: draw $bv \parallel hh'$, then v divides the weight line into two parts, which are the vertical resistances of the abutments. From v_1 draw $v_1o \parallel kk'$, then the closing line of the polygon e has the direction kk'.

A single joint at any point of an unsymmetrical arch can be treated in a similar manner.

A thrust produced by temperature strains will be applied along the closing line kk', and the bending moments induced will be proportional to the ordinates of the polygon d from this closing line. The variation of span must be computed not for the horizontal span, but for the projections of it on the closing line kk'. The construction of this component of the total effect will be like that previously employed. Another effect will be caused in a line perpendicular to kk'. The variation of span for this construction, is the projection of the total horizontal variation on a line perpendicular to kk', and the bending moments induced by this force applied at b_6, and perpendicular to the closing line, will be proportional to the horizontal distances of the points of division from b_6. As these constructions are readily made, and the shearing and tangential stresses determined from them, it is not thought necessary to give them in detail.

CHAPTER VIII.

ARCH RIB WITH TWO JOINTS.

Let us take the two joints, one at the center and one at one end as represented in Fig. 9. Let the loading, etc., be as in Fig. 6.

The closing line evidently passes through the two joints, as at them the bending moment vanishes.

The remaining condition to be fulfilled is that the deflection of the right half of the arch in the direction of this line, shall be the same as that of the left half.

Let us then suppose that the straight girder $b_6'p'$ perpendicular to the closing line, is fixed at b_6' and bent first by the moments M_a giving us the deflection curve $b_6'f'$ when b_6' is taken as the pole, and the loads of the type mm are one-quarter of the corresponding ordinates of the polygon d; and secondly, by the moments M_c giving us the deflection curve $b_6'g'$ when drawn with the same pole, and the loads of the type nn also one-quarter of the corresponding ordinates of the polygon c. It should be noticed that the points at which these moments are supposed to be concentrated in the girder $b_6'p'$, are on the parallels to kk' through the points d_5, d_6, etc.

Similarly let ff_3 and f_3f_6 be the deflection curves of the straight girder d_3p (using d_3 as the pole distance), under the applied moments.

We have used now a pole distance differing from that used in the right half of the arch. These pole distances must have the same ratio that the quantity EI has for the two parts of arch. If EI is the same in both parts of the arch the same pole distance must be used to obtain the deflection curves in both sides of the middle. In the same manner the curves gg_3 and g_3g_6 are found. Now must the moments M_c causing the total deflection $p'g'-gg_6=\frac{1}{2}ai$ be elongated so that they shall cause a total deflection $pf'-ff_6=\frac{1}{2}aj$. The ratio lines ai_0, aj_0' will enable us to find the new position t_2 of the load line to effect this.

To find o the new pole, through v_2, which divides the load line into parts which are the vertical resistances of the piers, draw $v_2o \parallel b_6k$. Then draw the polygon e as in Fig. 7, starting from d. It must pass through b_6. We can find also whether ke_6' has the required ratio to hc_6' by the aid of the ratio lines, which will further test the accuracy of the work.

Any unsymmetrical arch with joints situated differently from the case considered can be treated by a like method.

The temperature strains should be treated like those in Fig. 8, which are caused by a thrust along the closing line. Those at right angles to this line vanish as the joints allow motion in this direction. The shearing and tangential stresses can be found as in Fig. 3.

Arches with more than three hinge joints are in unstable equilibrium, and can only be used in an inverted position as suspension bridges. These will be treated subsequently. If the joints, however, possess some stiffness so that they are no longer hinge joints, but are block-work joints, or analogous to such joints, we may still construct arches which are stable within certain limits although the number of joints is indefinitely increased. Such are stone or brick arches. These will also be treated subsequently.

The constructions in Figs. 6, 7, 8, 9, can be tested by a process like that employed in Figs. 2 and 3. In Fig. 2, for instance, we obtained the algebraic sum of the squares of the quantities of the type *ss*, and showed that such sum vanishes. We can obtain the same result in all cases.

CHAPTER IX.

THE CINCINNATI AND COVINGTON SUSPENSION BRIDGE. (Fig. 10.)

The main span of this bridge has a length of 1057 feet from center to center of the towers, and the end spans are each 281 feet from the abutment to the center of the tower. The deflection of the cable is 89 feet at a mean temperature, or about 1—11.87th of the span. There is a single cable at each side of the bridge. Each of these cables is made up of 5200 No. 9 wires, each wire having a cross-section of 1–60th of a square inch and an estimated strength of 1620 lbs. Each of these cables has a diameter of 12½ inches, and an estimated strength of 4212 tons. Each cable rests at the tower upon a saddle of easy curvature, the saddle being supported by 32 rollers which run upon a cast iron bed-plate 8×11 feet, which forms part of the top of the tower. Since the bed-plate is horizontal this method of support ensures the exact perpendicularity of the force which the cables exert upon the towers, without its being necessary to make the inclination of the cable on both sides of the saddle the same. There is, therefore, no tendency by the cables to overturn the towers, and they need only be proportioned to bear the vertical stresses coming upon them.

As this bridge differs greatly in some respects from other suspension bridges, it seems necessary to describe its peculiarities somewhat minutely.

The roadway and sidewalks make a platform 36 feet wide, extending from abutment to abutment, 1619 feet. It is built of three thicknesses of plank solidly bolted together, in all 8 inches thick. This is strengthened by a double line of rolled I girders, 1630 feet long, running the entire length of the center of the platform. These I girders are arranged one line above the other, and across between them, at distances of 5 feet, run lateral I girders which are suspended from the cable. The upper line of girders is 9 inches deep, (and 30 lbs. per foot); the lower line is 12 inches deep (and 40 lbs. per foot). The lateral girders are 7 inches deep (and 20 lbs. per foot), and are firmly embraced between the double line of longitudinal girders. The girders of this center line are each 30 ft. long, and are spliced together by plates in the hollows of the I, but the holes through which the bolts pass are slots whose length is two or three times the diameter of the bolts. This makes a "slip joint" such as is often used in fastening the ends of the rails on a railroad. The slip joints permit the wooden planking of the roadway to expand and contract from variations of moisture and temperature without interference from the iron girders which are bolted to it.

There is also a line of wrought-iron truss-work about 10 feet deep extending from abutment to abutment on each side of the roadway, consisting of panels of 5 feet each, to each lower joint of which is fastened a lateral girder and a suspender from the cable. This trussing is a lattice, with vertical posts, and ties extending across two panels, and its chords are both made with slip joints every 30 feet.

It is apparent that this whole arrangement of flooring with the girders and trusses attached to it possesses a very small amount of stiffness, in fact the stiffness is principally that of the flooring itself. It will permit a very large deflection, say 25 feet, up or down from its normal position without injury. Its office is something quite different from that of the ordinary stiffening truss of a suspension bridge. It certainly serves to distribute concentrated loads over short distances, but not to the extent required, if that were the sole means of preserving the cable in a fixed position under the action of moving loads. Its true function is to destroy all vibrations and undulations, and prevent their propagation from point to point by the enormous frictional resistance of these slip joints. When a wave does work against elastic forces, the reaction of those forces returns the wave with nearly its original intensity, but when it does work against friction it is itself destroyed.

The means relied on in this bridge to resist the effect of unbalanced loads is a system of stays extending from the top of the tower in straight lines to those parts of the roadway which would be most deflected by such loads. There are 76 such stays, 19 from the top of each tower. The longest stays extend so far as to leave only 350 feet., *i.e.*, a little over one-third of the span, in the center over which they do not extend. Each stay being a cable 2¼ inches in diameter has an estimated strength of 90 tons. They are attached every 15 feet to the roadway at the lower joints of the trussing, and are kept straight by being fastened to the suspenders where they cross them. This system is shown in Fig. 10 in which all the stays for one cable are drawn, together with every third suspender. The suspenders occur every 5 feet throughout the bridge but none are shown in the figure except those attached at the same points as the stays.

These stays must sustain the larger part of any unbalanced load, at the same time producing a thrust in the roadway against either the abutment or tower.

It is really an indeterminate question as to how the load is divided between the stays and trussing; and this the more, because of the manner in which the other extremities of the stays are attached. Of the nineteen stays carried to the top of one tower, the eight next the tower are fastened to the bed plate under the saddle, and so tend to pull the tower into the river; the remaining eleven are carried over the top of the tower, and rest on a small independent saddle, beside the main saddle, and are eight of them fastened to the middle portion of the side spans as shown in Fig. 10, while the other three are anchored to the abutment.

In view of the indeterminate nature of the problem, it has seemed best to suppose that the stays should be proportioned to bear the whole of any excess of loading of any portion of the bridge, over the uniformly distributed load (which latter is of course borne by the cable itself); and further that the truss really does bear some fraction of the unbalanced load, and that the bending moments have therefore the same relative amounts as if they sustained the entire unbalanced load. This fraction, however, is quite unknown owing to the impossibility of finding any approximate value of the moment of inertia I for the combined wood and iron work of the roadway.

This method of treatment has for our present purpose this advantage, that the construction made use of is the same as that which must be used when there are no stays at all, and the entire bending moments induced by the live loads are borne by the stiffness of the truss alone.

Now in order to determine the tension in any stay, as for instance that in the longest stay leading to the right hand tower, lay off v_1v_2 equal to the greatest unbalanced weight, which under any circumstances is concentrated at its lower extremity. This weight is sustained by the longitudinal resistance of the flooring, and the tension of the stay. The stresses induced in the stay and flooring by the weight, are found by drawing from v_1 and v_2 the lines v_1o and v_2o parallel respectively to the stay and the flooring. Then v_1o is the tension of the stay, and that of the other stays may be found in a similar manner.

It is impossible to determine with the same certainty how the stress ov_2 parallel to the flooring is sustained. It may be sustained entirely by the compression it produces in the part of the flooring between the weight and the tower or the abutment; or it may be sustained by the tension produced in the flooring at the left of the weight; or the stress ov_2 may be divided in any manner between these two parts of the flooring, so that v_1v_1' may represent the tension at the left, and ov_2' the compression at the right of the weight. It appears most probable that the induced stress is borne in the case before us by the compression of the flooring at the right, for the flooring is ill suited to bear tension both from the slip joints of the iron work and the want of other secure longitudinal fastenings; but on the contrary it is well designed to resist compression. The flooring must then be able at the tower to resist the sum of the compressions produced by all the unbalanced weights which can be at once concentrated at the extremities of the nineteen stays.

There is one considerable element of stiffness which has not been taken account of in this treatment of the stays, which serves very materially to diminish the maximum stresses to which they might otherwise be subjected. This is the intrinsic stiffness of the cable itself which is formed of seven equal subsidiary cables formed into a single cable, by placing six of them around the seventh central cable, and enclosing the whole by a substantial wrapping of wire, so that the entire cable having a diameter of 12½ inches, affords a resistance to bending of from one sixth to one half that of a hollow cylinder of the same diameter and equal cross section of metal. Which of these fractions to adopt depends somewhat on the tightness and stiffness of the wrapping.

It is this intrinsic stiffness of the cable which is largely depended upon in the central part of the bridge, between the two longest stays, to resist the distortion caused by unbalanced weights.

As might be foreseen the distortions are actually much greater in the central part of the bridge than elsewhere, though they would have been by far the greater in those parts of the bridge where the stays are, had the stays not been used.

The center of a cable is comparatively stable while it is undergoing quite considerable oscillations, as may be readily seen by a simple experiment with a rope or chain.

Let us now determine the relative amount of the stresses in the stiffening truss, on the supposition that the actual stresses are some unknown fraction of the stresses which would be induced, if there were no stays, and the truss was the only means of stiffening the cable. We, therefore, have to determine only the total stresses, supposing there are no stays, and then divide each stress obtained by n (at present unknown) to obtain the results required. Let us draw the equilibrium polygon d which is due to a uniform load of depth xy, and which has a deflection bd six times the central deflection of the cable. The loading of the cable is so nearly uniform, that each of the ordinates of the type bd, may be considered with sufficient accuracy to be six times the corresponding ordinate of the cable. Any multiple other than six might have been used with the same facility. In order to cause the polygon to have the required deflection with any assumed pole distance it is necessary to assume the scale of weights in a particular manner, which may be determined easily in several ways. Let us find it thus :

Let W=one of the concentrated weights.
Let D=central deflection of cable.
Let S=span of the bridge
Let M=central bending moment due to the applied weights.

Then, if the pole distance$=\frac{1}{3}S$, $M=\frac{1}{3}S\times 6D=2SD$, for the moment is the product of the pole distance by the ordinate of the equilibrium polygon. Again, computing the central moment from the applied forces,

$$M=\tfrac{11}{2}W\times\tfrac{1}{2}S-5W\times\tfrac{1}{4}S=\tfrac{3}{2}WS,$$

in which the first term of the right hand member is moment of the resistance of the piers, and the second term is the moment of the concentrated weights applied at their center of gravity.

$$\therefore\ \tfrac{3}{2}WS=2SD\ \therefore\ W=\tfrac{4}{3}D,$$

Hence, if one-third of the span is to represent the pole distance or true horizontal tension of an equilibrium curve having six times the deflection of the cable, each concentrated weight when the span is divided into twelve equal parts, is represented by a length equal to $\frac{4}{3}$ of the deflection of the cable. The true horizontal tension of the cable will be six times that of the equilibrium polygon, or it will be represented, in the scale used, by a line twice the length of the span. Now taking b as the pole, at distances $bb_4 = bb_4' = \frac{1}{3}S$, lay off $b_4'w_1 = b_4w_1' = \frac{1}{2}W = \frac{2}{3}D$, so that they together represent the weight concentrated at b; and let $w_1w_2 = W$, represent the weight concentrated at b_2, etc. Then can the equilibrium polygon d be constructed by making $dd_1 \parallel bw_1$, $d_1d_2 \parallel bw_2$, etc. If $bd = 6D$ the polygon must pass through b_6 and b_6', which tests the accuracy of the work.

Now to investigate the effect of an unbalanced load covering one-half the span, let us take one half the load on the right half of the span and place it upon its left, so that xz and xb represent the relative intensity of the loading upon the left and right half of the span respectively, the total load being the same as before. If it is desirable to consider that the total load has been increased by the unbalanced load we have simply to change the scale so that the same length of load line as before, (viz, $b_4'w_6 + b_4w_6'$) shall represent the total loading. This will give a new value to the horizontal tension also.

Now let a new equilibrium polygon c be drawn, which is due to the new distribution of the concentrated weights. It is necessary to have the closing line of this polygon c horizontal, and this may be accomplished either, by drawing the polygon in any position and laying off the ordinates of the type bc equal to those in the polygon so drawn, or better as is done in this Figure by laying off in each weight line that part of the total load which is borne by each pier, which is readily computed, as follows. The distance of the center of gravity of the loading divides the span in the ratio of 17 to 27. Hence $\frac{27}{44}$ and $\frac{17}{44}$ of the total load are the resistances of the piers, or since the total load $= 11\,W$, we have $b_4'u_6 = \frac{27}{4}W$ and $b_4u_6' = \frac{17}{4}W$. Now make $u_6u_5 =$ the weight concentrated at b_5, etc., and $b_4'u_2 + b_4u_1 =$ that at b_1. Then draw the polygon c.

The polygon c has the same central deflection as the polygon d; for compute as before,

$$\therefore\ M = \tfrac{17}{4}W \times \tfrac{1}{2}S - \tfrac{5}{2}W \times \tfrac{1}{4}S = \tfrac{3}{2}\,WS$$

in which the first term of the second member is the moment of the resistance of the right pier, and the second term is the moment of the concentrated weights applied at their center of gravity.

By similar computations we may prove the following equalities;

$$d_5c_5 = d_1c_1 - -d_1'c_1' = -d_5'c_5';$$
$$d_4c_4 = d_2c_2 = -d_2'c_2' = -d_4'c_4';$$
$$d_3c_3 = -d_3'c_3'.$$

The quantities of the type dc are proportional to the bending moments which the stiffening truss must sustain if it preserves the cable in its original shape, when acted on by an unbalanced load of depth bx, on the supposition that the truss has hinge joints at its ends, and is by them fastened to the piers. For in that case the cable is in the condition of an arch with hinge joints at its ends. The condition which then holds is this:

$$\Sigma(M_d y) = \Sigma(M_c y)$$

or,

$$\Sigma(M_d - M_c)\,y = 0\ \therefore\ \Sigma(cd)\,y = 0.$$

This last is fulfilled as is seen by the above equations, for to every product such as $+b_1d_1 \times d_1c_1$ corresponds another $-b_1'd_1' \times d_1'c_1'$ of the same magnitude but opposite sign.

The polygon c could have been obtained by a second equilibrium polygon in a manner precisely like that used before, but as it appears useful to show the connection between the methods of treating the arch rib which is itself stiff, and the flexible arch or cable, which is stiffened by a separate truss, we have departed from our previously employed method for determining the polygon c, as it is easy to do when both c and d are parabolic.

Now let us compute the bending moment

$$= d_5c_5 \times \tfrac{1}{3}S = M^c - M_d$$
$$M_c = \tfrac{27}{4}W \times \tfrac{1}{12}S = \tfrac{9}{16}\,WS$$
$$M_d = \tfrac{11}{2}N \times \tfrac{1}{12}S = \tfrac{11}{24}\,WS$$
$$\therefore\quad M_c - M_d = \tfrac{5}{48}\,WS.$$

Compute also the bending moment at the vertical through b_4,

$$M_c = \tfrac{27}{4}W \times \tfrac{1}{6}S - \tfrac{3}{2}W \times \tfrac{1}{12}S = WS$$
$$M_d = \tfrac{11}{2}W \times \tfrac{1}{6}S - W \times \tfrac{1}{12}S = \tfrac{5}{6}\,WS$$
$$\therefore\ M_c - M_d = \tfrac{1}{6}\,WS$$

Similar computations may be made for the remaining points, and this noteworthy result will be found true, that the bending moments induced in the stiffening truss by the assumed loading, are the same as would have been induced by a positive loading on the left of a depth *yz*, and a negative loading on the right of an equal depth *yb*. For compute the moments due to such loading at the points b_5 and b_4.

The resistance of the pier due to such loading $= \frac{1}{4} W$

$$\therefore M_5 = \frac{1}{4} W \times \frac{1}{12} S = \frac{1}{48} WS$$

and

$$M_4 = \frac{1}{4} W \times \frac{1}{6} S - \frac{1}{2} W \times \frac{1}{12} S = \frac{1}{6} WS, \text{ etc.}$$

We arrive then at this conception of the stresses to which the stiffening truss is subjected, viz:—the truss is loaded with the applied weights acting downward, and is drawn upward by a uniformly distributed negative loading, whose total amount is equal to the positive loading, so that the load actually applied at any point may be considered to be the algebraic sum of the two loads of different signs which are there applied. This conception might have been derived at once from a consideration of the fact that the cable can sustain only a uniform load, if it is to retain its shape; but it appears useful in several regards to show the numerical agreement of this statement with Prop. IV of which in fact it is a particular case. It is unnecessary to make a general proof of this agreement, but instead we will now state a proposition respecting stiffening trusses, the truth of which is sufficiently evident from considerations previously adduced.

Prop. VI. The stresses induced in the stiffening truss of a flexible cable or arch, by any loading, is the same as that which would be induced in it by the application to it of a combined positive and negative loading distributed in the following manner, viz : the positive loading is the actual loading, and the negative loading is equal numerically to the positive loading, but is so distributed as to cause no bending moments in the cable or arch, *i.e.*, the cable or arch is the equilibrium polygon for this negative loading.

By flexible cable or arch is meant one which has hinge joints at the points where it supports the stiffening truss. It need not actually have hinge joints at these points : the condition is sufficiently fulfilled if it is considerably more flexible than the truss which it supports.

The truth of Prop. VI has been recognized by previous writers upon this subject in the particular case of the parabolic suspension cable, and it has been erroneously applied to the determination of the bending moments in the arch rib in general. It is inaccurate for this purpose in two particulars, inasmuch as in the first place the arch to which it is applied is not parabolic, though the negative loading due to it is assumed to be uniform, and in the second place the horizontal thrust is not the same for the different kinds of arch rib, while this assumes the same thrust for all, viz : that arising from a flexible arch or one with three or more joints.

A similar proposition has been introduced into a recent publication on this subject*, but in that work the truss stiffens a simple parabolic cable, and the truss is not supposed to be fastened to the piers, so that it may rise from either pier whenever its resistance becomes negative. As this should not be permitted in a practical construction the case will not be discussed. In accordance with Prop. VI let us determine anew the bending moments due to an unbalanced load on the left of an intensity denoted by *bz*. As before seen this produces the same effect as a positive loading of an intensity $yz = fm = \frac{1}{2} bz$ on the left, and a negative loading of an intensity $yb = fn = \frac{1}{2} bz$. Now using g as a pole with a pole distance of gf_2=one third of the span lay off the concentrated weight $p_1 p_2$=that applied at b_1, etc., on the same scale as the weights were laid off in the previous construction, and in such a position that g is opposite the middle of the total load, which will cause the closing line to be horizontal. Then draw the equilibrium polygon a due to these weights. The ordinates of the type *af* are by Prop. VI proportional to the bending moments induced in the stiffening truss by the unbalanced load when the truss is simply fastened to the piers at the ends, and, as we have seen, each of the quantities *af* is identical with the corresponding quantity *cd*.

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* Graphical Statics, A. J. Du Bois, p. 329, published by John Wiley & Son, New York.

If the stiffening truss is fixed horizontally at its ends a closing line *hh'* must be drawn in such a position that $\Sigma(M)=0$, and as it is evident that it must divide the equilibrium polygon symmetrically it passes through f its central point.

As stated in a previous article, the maximum bending moments at certain points of the span are caused when the unbalanced load covers somewhat more than half of the span. In the case of a parabolic cable or arch the maximum maximorum bending moment is caused when this load extends over two-thirds of the span, as is proved by Rankine in his Applied Mechanics by an analytic process. Let the load extend then over all except the right hand third of the span with an intensity represented by $bz=q_3q_4'$. Then if $f_2'q_3=\frac{1}{2}f_2'q_4'$, the truss may by Prop. VI be considered to sustain a positive load of the intensity $f_2'q_3$ on the left of b_2', and a negative load of the intensity $f_2'q_4'$ on the right of b_2'. Using g' as the pole and the same pole distance as before, lay off the weight q_6q_5 concentrated at b_5, etc., so that g' is opposite the middle of the weight line. We thus obtain the equilibrium polygon e, in which the ordinates of the type *ef* are proportional to the bending moments of the truss under the assumed loading, when its ends are simply fastened to the piers.

Now *bd* was the ordinate of an equilibrium polygon having the same horizontal tension, and under a load of the same intensity covering the entire span. It will be found that $bd=\frac{27}{4}f_2e_2$, which may be stated thus:—the greatest bending moment induced in the stiffening truss, by an unbalanced load of uniform intensity is four twenty-sevenths of that produced in a simple truss under a load of the same intensity covering the entire span. This result was obtained by Rankine analytically. If the truss is fixed horizontally at its ends, we must draw a closing line *kk'*, which fulfills the conditions before used for the straight girder fixed at the ends, as discussed previously in connection with the St. Louis Arch. By the construction of a second equilibrium polygon, as there given, we find the position of *kk'*; then the ordinates *ke* will be proportional to the bending moments of the stiffening truss.

The shearing stress in the truss is obtained from the loading which causes the bending moment, in the same manner as that in any simple truss. The horizontal tension in the cable, is the same whenever the total load on the span is the same, and is not changed by any alteration in the distribution of the loading, which fact is evident from Prop. VI. The maximum tension of the cable is found when the live load extends over the entire span, and is to be obtained from a force polygon which gives for its equilibrium polygon the curve of the cable itself, as would be done by using the weights w_1w_2, etc., and a pole distance of six times bb_4=twice the span.

The temperature strains of a stiffening truss of a suspension bridge are more severe than those of the truss stiffening an arch, because the total elongation of the cable in the side spans as well in the main span, is transmitted to the main span and produces a deflection at its center. This is one reason why stays furnish a method of bracing, particularly applicable to suspension bridges. But supposing that the truss bears part of the bending moment due to the elongation of the cable, it is evident that when the truss is simply fastened to the piers, the bending moments so induced are proportional to the ordinates of the type *bd*, for by the elongation of the cable, it transfers part of its uniformly distributed weight to the truss.

That load which the cable still sustains, is uniformly distributed, if the cable still remains parabolic, therefore that transferred to the truss is uniformly distributed.

When the truss is fixed horizontally at the piers, the closing line of the curve d must be changed so that $\Sigma(M)=0$, and the bending moments induced by variations of temperature, will be proportional to the ordinates between the curve d and this new closing line.

It remains only to discuss the stability of the towers and anchorage abutments. The horizontal force tending to overturn the piers comes from a few stays only, as was previously stated, and is of such small amount that it need not be considered.

The weight of the abutment in the case before us is almost exactly the same as the ultimate strength of the cable. Suppose that $st=sv$ are the lines representing these quantities in their position relatively to the abutment. Since their resultant sv intersects the base beyond the face of the abutment, the abutment would tip over before the cable could be torn asunder. And since the angle vsr is greater than the angle of friction between the abutment and the ground it stands on, the abutment if standing on the surface of the ground, would slide before the cable could be torn asunder.

The smallest value which the factor of safety for the cable assumes under a maximum loading is computed to be six. Take $st'=\frac{1}{6}st$ as the greatest tension ever induced in the cable, then sr' the resultant of sv and st' cuts the base so far within the face that it is apparent that the abutment has sufficient stability against overturning, and the angle vsr' is so much smaller than the least value of the angle of friction between the abutment and the earth under it, that the abutment would not be near the point of sliding even if it stood on the surface of the ground. It should be noticed tha all the suspenders in the side span assist in reducing the tension of the cable as we approach the abutment, and conduce by so much to its stability. Also the thrust of the roadway may assist the stability of the abutment, both with respect to overturning and sliding.

CHAPTER X.

THE CONTINUOUS GIRDER WITH VARIABLE CROSS-SECTION.

In the foregoing chapters the discussion of arches of various kinds has been shown to be dependent upon that of the straight girder; but as no graphical discussion has, up to the present time, been published which treats the girder having a variable cross-section and moment of inertia, our discussion has been limited to the case of arches with a constant moment of inertia.

Certain remarks were made, however, in the first chapter tending to show the close approximation of the results in case of a constant moment of inertia to those obtained when the moment of inertia is variable. We, in this chapter, propose a new solution of the continuous girder in the most general case of variable moment of inertia, the girder resting on piers having any different heights consistent with the limits of elasticity of the girder. This solution will verify the remarks made, and enable us easily to see the manner in which the variation of the moment of inertia affects the distribution of the bending moments, and by means of it the arch rib with variable moment of inertia can be treated directly.

Besides the importance of the continuous girder in case it constitutes the entire bridge by itself, we may remark that the continuous girder is peculiarly suited to serve as the stiffening truss of any arched bridge of several spans in which the arches are flexible. Indeed, it is the conviction of the writer that the stiff arch rib adopted in the construction of the St. Louis Bridge was a costly mistake, and that, if a metal arch was desirable, a flexible arch rib with stiffening truss was far cheaper and in every way preferable.

Let us write the equation of deflections in the form

$$mD\,.\,\frac{EI_0}{mn^2n'}=\Sigma\left(\frac{Mi}{nn'}\cdot\frac{x}{n}\right)$$

in which n is the number by which any horizontal dimension of the girder must be divided to obtain the corresponding dimension in the drawing, n' is the divisor by which force must be divided to obtain the length by which it is to be represented in the drawing, m is an arbitrary divisor which enables us to use such a pole distance for the second equilibrium polygon as may be most convenient, I_0 is the moment of inertia of the girder at any particular cross section assumed as a standard with which the values of I at other cross sections are compared, and $i=I_0\div I$ is the ratio of I_0 (the standard moment of inertia), to I (that at any other cross-section). For the purpose of demonstrating the general properties of girders, the equation need not be encumbered with the coefficients mnn', but for purposes of explaining the graphical construction they are very useful, and can be at once introduced into the equation when needed.

In the equation

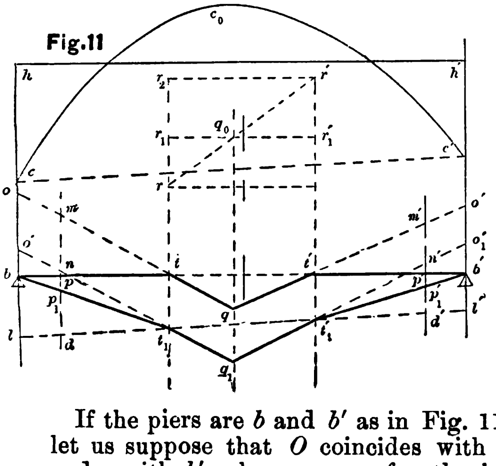
$$D\,.\,EI_0=\Sigma_a^o(Mix)$$

the quantity D is the deflection of any point O of the girder below the tangent at the point a where the summation begins, and M is the actual bending moment at any point between O and a. These moments M at any point consist in general of three quantities, represented in the construction by the positive ordinate of the equilibrium polygon due to the weights, and by the two negative ordinates of the triangles into which we have divided the negative moment area. If we distinguish these components of M by letting M_0 represent that due to the weights, while M_1 and M_2 represent the components due to the left and right negative areas respectively, the equation of deflections becomes

$$D\,.\,EI_0=\Sigma_a^o(M_0ix)-\Sigma_a^o(M_1ix)-\Sigma_a^o(M_2ix)$$

Now let us take O at a pier at one end of a span and extend the summation over the entire span.

Fig. 11



If the piers are b and b' as in Fig. 11, let us suppose that O coincides with b and a with b'; also suppose for the instant that I is constant, so that $i=1$ at all points of the girder. Then we have

$$D_b\,.\,EI=\bar{x}_5\,\Sigma_{b'}^{b}(M_0)-\bar{x}_1\,\Sigma_{b'}^{b}(M_1)$$
$$-\bar{x}_2\,\Sigma_{b'}^{b}(M_2)$$

in which D_b is the deflection of b below the tangent at b', $\bar{x}_0$ is the distance of the center of gravity of the moment area due to the applied weights from b, while $\bar{x}_1$ and $\bar{x}_2$ are the distances of the centers of gravity of the negative areas from b. In Fig. 11 let cc_0c' be the positive area due to the weights and representing $\Sigma_{b'}^{b}(M_0)$, while $\Sigma_{b'}^{b}(M_1)$ and $\Sigma_{b'}^{b}(M_2)$ are represented by hcc' and $hh'c'$ respectively. Let the center of gravity of cc_0c' be in qq_0, while the centers of the two negative areas are in tr and $t'r'$. Let the height of a triangle on some assumed base, and equivalent in area to cc_0c', be rr_2, then by a process like that in Fig. 2 it is evident that rr_1 and r_1r_2 are the heights of the right and left negative triangles, having the assumed base, on the supposition that the girder is fixed horizontally over the piers.

Now introducing the constants mnn' into the last equation and into the equation before that, the relation of the quantities is such that if the moments be applied as weights at their centers of gravity with the pole distance $pt=EI\div mn^2n'$, the equilibrium polygon so obtained will be tangent at the piers to the exaggerated deflection curve obtained when the distributed moments are used as weights; and the deflection at the pier b from the tangent at b' will be the same as that of this exaggerated deflection curve, and vice versa.

Let $pm=r_1r_2$, $p'm'=rr_1$ and $pt=p't$, then t and t' constitute the pole, pm and $p'm'$ the negative loads, and $pm+p'm'$ the positive load. Then is $btqt'b'$ the equilibrium polygon for these loads. The deflection of b below $b't'$ vanishes as it should in case the girder is fixed horizontally over the pier.

Now let the direction of the tangents at the piers be changed so that the tangents to the exaggerated deflection curve assume the directions bt_1 and $b't_1'$. Then the load line and force polygon assume a new position, such that t_1 and t_1' form the pole, and $dn=pm$ and $d'n'=p'm'$ comprise the positive load while np_1 and $n'p_1'$ are the new negative loads which will cause the equilibrium polygon $bt_1q_1t_1'b'$, which is due to them, to have its sides bt_1 and $b't_1'$ in the directions assumed.

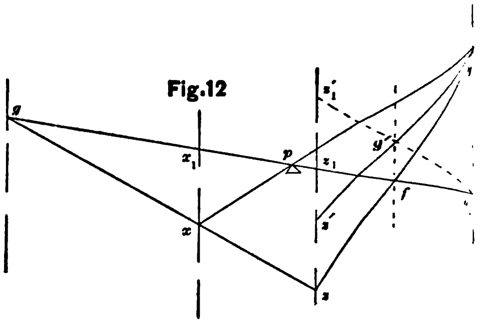
There are several relations of quantities in this figure to which we wish to direct attention. It is evident, in case I is not constant, that from the area cc_0c' whose ordinates are proportional to M_0, the actual bending moments due to the weights, another area whose ordinates are proportional to

M_0i, the effective bending moments, can be obtained by simple multiplication, since i is known at every point of the girder. Moreover, the vertical through the center of gravity of this positive effective moment area can be as readily found as that through the actual positive moment area. Call this vertical "the positive center vertical." Again, the negative moment areas proportional to M_1i and M_2i can be found from the triangular areas proportional to M_1 and M_2 by simple multiplication, and if we proceed to find the verticals through their centers of gravity we shall obtain the same verticals whatever be the magnitude of the negative triangular areas, since their vertical ordinates are all changed in the same ratio by assuming the negative areas differently. Let us call these verticals the "left" and "right" verticals of the span. In case $i=1$, as in Fig. 11, the left and right verticals divide the span at the one-third points. This matter will be treated more fully in connection with Fig. 13.

Again, let us call the line t_1t_1' "the third closing line." It is seen that, whatever may be the various positions of the tangent bt_1, the ordinate dn, between the third closing line and t_1q_1 prolonged, is invariable; for the triangle $t_1q_1t_1'$ is invariable, being dependent on the positive load and pole distance alone. By similarity of triangles it then follows that the ordinate, such as lo', on any assumed vertical continues invariable; and when there is no negative load at t_1, then bt_1q_1 becomes straight, o' coincides with b and n with p_1. Similar relations hold at the right of q_1. The quantity dp_1 is of the nature of a correction to be subtracted from the negative moment when the girder is fixed horizontally at the piers in order to find the negative moment when the tangent assumes a new position, for $np_1=dn-dp_1$. The negative moments can consequently be found from the third closing line and the tangents at the piers; while the remaining lines q_1t_1 and $q_1't_1'$ will test the correctness of the work. Before applying these properties of the deflection polygon and its third closing line to a continuous girder, it is necessary to prove a geometrical theorem from Fig. 12.

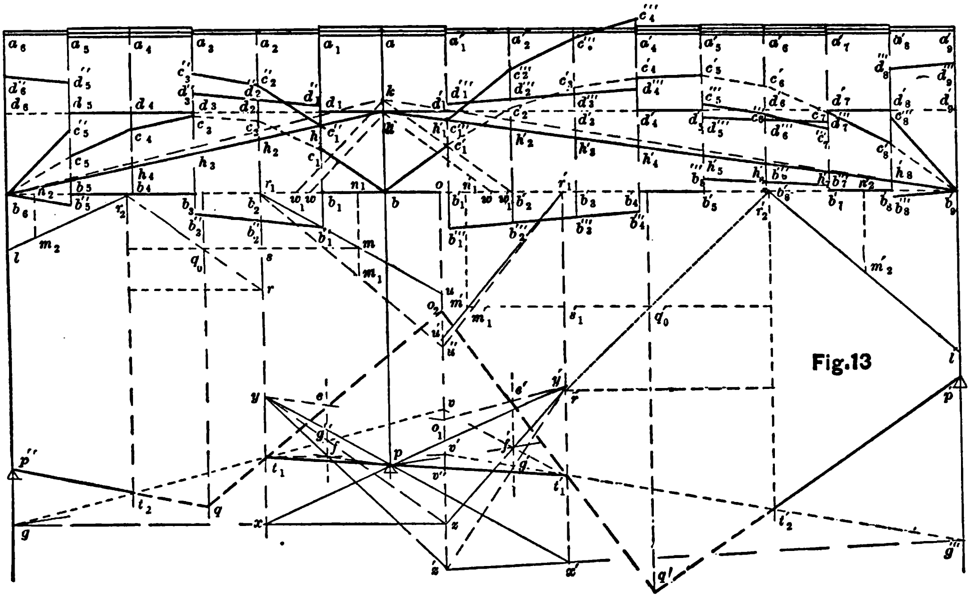
Let the variable triangle xyz be such that the side xz always passes through the fixed point g, the side xy always passes through the fixed point p, and the vertices xyz are always in the verticals through those points; then by the properties of homologous triangles the side yz also has a fixed point f in the straight line gp. Furthermore, if there is a point z' in the vertical through z, and in all positions of z it is at the same constant distance from z, then on the line yz' there is a fixed point g' where the vertical through f intersects yz'; for, if z' maintains its distance zz' invariable, then must any other point as g' remain constantly at the same vertical distance from f, as appears from similarity of triangles. But as f is fixed g' is also. When, for instance, the triangle xyz assumes the position $x_1y_1z_1$, then z' moves to z_1'.

Fig.12



Let us now apply the foregoing to the discussion of a continuous girder over three piers $p''pp'$ as shown in Fig. 13, in which the lengths of the spans have the ratio to each other of 2 to 3. Divide the total length of the girder into such a number of equal parts or panels, say 15, that one division shall fall at the intermediate pier, and let the number of lines in any panel of the type aa represent its relative moment of inertia. Assume the moment of inertia where there are three lines, as at a, a_4, etc., as the standard or I_0, then $i=1$ at a, $i=\frac{3}{2}$ at a_2, $i=\frac{3}{4}$ at a_6', etc.

Let the polygons c and c' be those due to the weights in the left and right spans respectively. Then the ordinates of the type bc are proportional to M_0 in the left span. The figure $bc_1c_1''c_2''c_3''c_3c_4c_5c_5''$ b_6 is the positive effective moment area in the left span, and its ordinates are proportional to M_0i. Its center of gravity has been found, by an equilibrium



polygon not drawn, to lie in the positive center vertical qq_0. A similar positive effective moment area on the right has its center of gravity in the positive center vertical $q'q_0'$.

Now assume any negative area, as that included between the lines *b* and *d*, and draw the lines hb_6 and hb_9', dividing the negative area in each span into right and left triangular areas. Let the quantities of the type *hb* be proportional to M_1, *hd* to M_2, *h'b'* to M_1', etc., then the ordinates of $bb_1b_1''b_2''b_3''b_3b_5b_5''b_6h$ are proportional to M_1i, and the center of gravity of this area has been found to lie in the right negative vertical t_1r_1. Similarly, the left negative vertical containing the center of gravity of the left negative effective moments, is t_2r_2. In the right span $t_1'r_1'$ and $t_2'r_2'$ are the left and right verticals. As before stated, these verticals would not be changed in position by changing the position in any manner whatever of the line *d* by which the negative moments were assumed, for such change of position would change all the ordinates in the same ratio.

Let us find also the vertical containing the center of gravity of the effective moment area, corresponding to the actual moment area b_6hb_9'. It is found by a polygon not drawn to be *vo*. Call *vo* "the negative center vertical." It is unchanged by moving the line *d*. If a polygon be drawn due to the effective moments as loads, two of its sides must intersect on *vo*, because it contains the center of gravity of contiguous loads. Now let rr_1 represent $\Sigma(M_0i)$:—it is in fact one eighth of the sum of the ordinates $b_1c_1+b_1c_1''$, etc., and hence is the height of a triangle having a base$=\frac{1}{4}bb_6$, and an area equal to the effective moment area in the left span. Also $r'r_1'$ is the height of a triangle having the same base, and an area equal to the effective moment area in the right span.

As previously explained, sr_1 is the amount of the right negative effective moment area in the left span, measured in the same manner, while *sr* is that on the left when the girder is fixed horizontally at the piers. We obtain $s'r_1'$ and $s'r'$ in the right span, in a similar manner. Now assume the arbitrary divisor $m=1$, and take the pole distance $r_1n_1=EI_0\div n^2n'$. Then as seen previously, if $mn_1=sr_1$, *ou* is the constant intercept on the negative center vertical, between the third closing line in the left span, and a side of the type *qt*. Also *ou'* is a similar constant intercept on this vertical due to the right span. Make $r_2n_2=r_1n_1$ and $n_2m_2=sr$, then lb_6 is a similar invariable intercept; as is $l'b_9'$, which is obtained in a similar manner.

Now the negative center vertical *ov* was obtained from the triangle b_6hb_9', *i.e.*

on the supposition that the actual moment over the pier is the same whether it be determined from the left or right of the pier. It is evident that while the girder is fixed horizontally at the intermediate pier, the moment at that pier is generally different on the two sides, at points infinitesimally near to it, but that when the constraint is removed an equalization takes place.

Since ou and ou' are derived from the positive effective moments, it appears that when the tangent at p is in such a position that the two third closing lines intercept a distance uu' on ov and the two lines of the type qt when prolonged intersect on ov, the moments over the pier will have become equalized.

We propose to determine the position of the tangent at p which will cause this to be true, by finding the proper position of the third closing lines in the two spans.

Move the invariable intercepts to a more convenient position, by making $o_1z = ou$, and $o_1z' = ou'$. Now by making the arbitrary divisor $m = 1$, as we did, the ordinates of the deflection polygon became simply D, *i.e.*, they are of the same size in the drawing as in the girder, hence the difference of level of p'', p and p' must be made of the actual size. By changing m this can be increased or diminished at will.

Now we propose to determine two fixed points g and g'', through which the third closing line in the left span must pass, and similarly g''' and g' on the right.

If the girder is free at p'' then as shown in connection with Fig. 11, the third closing line must pass through g, if $gp'' = lb_6$. Draw gz as a tentative position of the third closing line, and complete the triangle $xy'z$ as in Fig. 12.

Then is xy' the tentative position of the tangent at p, and since the third closing line in the right span must pass through y', and make an intercept on the negative center vertical equal to uu', then $z'y'$ is its corresponding tentative position. But wherever gz may be drawn, every line making an intercept $= uu'$ and intersecting $t_1'r_1'$ in such a manner that the tangent passes through p must pass through the fixed point g', found as described in Fig. 12. Therefore the third closing line in the right span passes through g'. Similarly, if there were more spans still at the right of these, we should use g' for the determination of another fixed point, as we have used g to determine it.

Now find g''' and g'' precisely as g and g' have been found, and draw the third closing lines t_1t_2 and $t_1't_2'$. If t_1t_1' passes through p the construction is accurate. Make $uu'' = vv''$, then is n_1m_1 the negative effective moment at the left, and $n_1'm_1'$ that at the right of the pier.

Let bw be the effective moment area corresponding to the triangle hbb_6, and measured in the same manner as the positive area was, by taking one eighth of its ordinates, and let $bw_1 = n_1m_1$; then as the effective moment bw is to the actual moment bh corresponding to it, so is the effective moment bw_1 or n_1m_1 to the actual moment bk corresponding to it. The same moment bk is also found from $n_1'm_1'$, by an analogous construction at the right of b, which tests the accuracy of the work.

Several other tests remain which we will briefly mention.

Prolong $p''t_2$ to q, and $p't_2'$ to q', then qt_1 and $q't_1'$ must intersect on the negative center vertical at o_2 so that $o_2v'' = ou''$. Also vv' must be equal to uu'. Again t_1v' passes through f, and $t_1'v$ through f'. Also yo_1 intersects qo_2 on the fixed vertical fg'' at e, and $y'o_1$ intersects $q'o_2$ on the fixed vertical $f'g'$ at e'. That these must be so is evident from a consideration of what occurs during a supposed revolution of the tangent t_1t_1', to the position xy'.

Now having determined the moment bk over the pier, kb_6 and kb_9' are the true closing lines of the moment polygons c and c'. Call these closing lines k, then the ordinates of the type kc will represent the bending moments at different points of the girder. The points of the contra flexure are at the points where the closing lines intersect the polygons c and c'. The directions of the closing lines will permit at once the determination of the resistances at the piers and the shearing stresses at any point.

The particular difference between the construction in case of constant and of variable moment of inertia, is seen to be in the positions of the center verticals positive and negative, and the right and left verticals.

The small change in their position due to the variation in the moment of inertia, is the justification of the remarks previously made respecting the close approximation of the two cases.

It is seen that the process here developed can be applied with equal facility to a girder with any number of spans. Also if the moment of inertia varies continuously instead of suddenly, as assumed in Fig. 13, the panels can be taken short enough to approximate with any required degree of accuracy to this case.

CHAPTER XI.

THE THEOREM OF THREE MOMENTS.

The preceding construction has been in reality founded on the theorem of three moments, but when the equation expressing that theorem is written in the usual manner, the relationship is difficult to see. Indeed the equation as given by Weyrauch* for the girder having a variable moment of inertia, is of so complicated a nature that it may be thought hopeless to attempt to associate mechanical ideas with the terms of the equation, in any clearly defined relationship. We propose to derive and express the equation in a novel manner, which will at once be easy to understand, and not difficult of interpretation in connection with the preceding construction.

Let us assume the general equation of deflections in the form.

$$D = \Sigma(Mx \div EI), \text{ or } D.EI_0 = \Sigma(Mix) \qquad (7)$$

in which I is the variable moment of inertia, I_0 some particular value of I assumed as the standard of comparison, $i = I_0 \div I$, and x is measured horizontally from the point as origin, where the deflection D is taken to the point of application of the actual bending moment M. The quantity Mi is called the effective bending moment, and the deflection D is the length of the perpendicular from the origin to the line tangent to the deflection curve at point to which the summation is extended.

Now consider two contiguous spans of a continuous girder of several spans, and let acb denote the piers, c being the intermediate pier. Let the span $ac = l$ and $bc = l'$. Take the origin at a and extend the summation to c, calling the deflection at a, D_a. When the origin is at b and the summation extends to c, let the deflection be D_b. Let also y_a, y_b and y_c be the heights of a, b and c respectively above some datum level. Then, as may be readily seen,

$$D_a = y_a - y_c - lt_c\ ,$$
$$D_b = y_b - y_c - l't_c',$$

if t_c is the tangent of the acute angle at c on the side towards a between the tangent line of the deflection curve at c and the horizontal, and t_c' is the tangent of the corresponding acute angle on the side of c towards b.

Now if we consider equation (7) to refer to the span l, the moment M may be taken to be made up of three parts, viz:—M_0 caused by the weights on the girder, M_1 dependent on the moment M_c at c, and M_2 dependent on the moment M_a at a. The moments in the span l' may be resolved in a similar manner. We may then write the equations of deflections in the two spans when the summation extends over each entire span as follows:

$$EI_0(y_a - y_c - lt_c) = \Sigma_c^a(M_0ix) - \Sigma_c^a(M_1ix)$$
$$- \Sigma_c^a(M_2ix) \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

$$EI_0(y_b - y_c - l't_c') = \Sigma_c^b(M_0'i'x')$$
$$- \Sigma_c^b(M_1'i'x') - \Sigma_c^b(M_2'i'x') \qquad (9)$$

in which x is measured from a, and x' from b towards c. Now if the girder is originally straight, $t_c = -t_c'$, hence we can combine these two equations so as to eliminate t_c and t_c', and the resulting equation will express a relationship between the heights of the piers, the bending moments (positive and negative), their points of application and the moments of inertia; of which quantities the negative bending moments are alone unknown. The equation we should thus obtain would be the general equation of which the ordinary expression of the theorem of three moments is a particular case. Before we write this general equation it is desirable to introduce certain modifications of form which do not diminish its generality. Suppose that

$$\bar{x}_1 \Sigma_c^a(M_1i) = \Sigma_c^a(M_1ix)$$

then is $\bar{x}_1$ the distance from a to the center of gravity of the negative effective moment area next to c. As was shown in connection with Fig. 13, the position of this center of gravity is independent of the magnitude of M_1 or M_c and may be found from the equation,

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* Allgemeine Theorie und Berechnung der Continuirlichen und Einfachen Trager. Jakob I. Weyrauch. Leipzig 1873.

$$\bar{x}_1 = \frac{\int_c^a ix^2dx}{\int_c^a ixdx} \quad . \quad . \quad . \quad . \quad (10)$$

for M_1 is proportional to x. Similarly it may be shown that

$$\bar{x}_2 = \frac{\int_c^a i(l-x)xdx}{\int_c^a i(l-x)dx} \quad . \quad . \quad (11)$$

is the distance of the center of gravity of the negative effective moment area next to a.

Again, suppose that

$$i_1\Sigma_c^a(M_1) = \Sigma_c^a(Mi)$$

then is i_1 an average value of i for the negative effective moment area next to c, which is likewise independent of the magnitude of M_1, as appears from reasoning like that just adduced respecting $\bar{x}_1$. Hence i_1 may be found from the equation

$$i_1 = \frac{\int_c^a ixdx}{\int_c^a xdx} \quad . \quad . \quad . \quad (12)$$

Similarly it may be shown that

$$i_2 = \frac{\int_c^a i(l-x)dx}{\int_c^a (l-x)dx} \quad . \quad . \quad (13)$$

in which i_2 is the average value of i for the negative effective moment area next to a.

The integrals in equations (10), (11), (12), (13), and in others like them referring to the span l', which contain i must be integrated differently, in case i is discontinuous, as it usually is in a truss, from the case where i varies continuously. When i is discontinuous the integral extending from c to a must be separated into the sum of several integrals, each of which must extend over that portion of the span l in which i varies continuously.

Furthermore we have

$$\Sigma_c^a(M_1) = \tfrac{1}{2}M_cl \quad . \quad . \quad . \quad (14)$$

since each member of this equation represents the negative actual moment area next to c in the span l.

Similarly, we have the equations

$$\Sigma_c^a(M_2) = \tfrac{1}{2}M_al\ , \quad \Sigma_c^b(M_1') = \tfrac{1}{2}M_c'l',$$
$$\Sigma_c^b(M_2'l') = \tfrac{1}{2}M_bl'.$$

If there is no constraint at the pier then must $M_c = M_c'$.

Now making the substitutions in equations (8) and (9), which have been indicated in the developments just completed, and then eliminating t_c and t_c',

$$EI_0\left\{\frac{y_a-y_c}{l} + \frac{y_b-y_c}{l'}\right\} - \frac{\bar{x}_0 i_0}{l}\Sigma_c^a(M_0) - \frac{\bar{x}_0' i_0'}{l'}\Sigma_c^b(M_0') = \tfrac{1}{2}[M_a\bar{x}_2i_2 + M_c(\bar{x}_1i_1 + \bar{x}_1'i_1') + M_b\bar{x}_2'i_2'] \quad . \quad . \quad . \quad (15)$$

in which $\bar{x}_0$ is the distance from a of the center of gravity of the positive effective moment area due to the weights in the span l, and $\bar{x}_0'$ is a similar distance from b in the span l', while i_0 and i_0' are average values of i for these areas derived from the equations in each span,

$$i_0 = \Sigma(M_0i) \div \Sigma(M_0).$$

It may frequently be best to leave the expressions containing the positive moments in their original form as expressed in equations (8) and (9).

Equation (15) expresses the theorem of three moments in its most general form.

Let us now derive from equation (15), the ordinary equation expressing the theorem of three moments, for a girder having a constant cross section. In this case $i=1$, and we wish to find the value of the term $\Sigma(M_0x)$ in each span. Let M_0 be caused by several weights P applied at distances z from a, then the moment due to a single weight P at its point of application is

$$M_z = Pz(l-z) \div l,$$

which may be taken as the height of the triangular moment area whose base is l which is caused by P. This triangle whose area is $\frac{1}{2}M_zl$ is the component of $\Sigma(M_0)$ due to P and can be applied as a concentrated bending moment at its center of gravity at a distance x from a.

Now $x = \frac{1}{3}(l+z)$, and taking all the weights P at once

$$\Sigma^a(M_0x) = \tfrac{1}{6}\Sigma_c^a[P(l^2-z^2)z].$$

Also in equation (15) we have in this case

$$\bar{x}_1=\tfrac{1}{3}l,\ \ \bar{x}_2=\tfrac{2}{3}l,\ \ \bar{x}_1{}'=\tfrac{1}{3}l',\ \ \bar{x}_2{}'=\tfrac{2}{3}l'$$

$$\therefore\ 6EI\left\{\frac{y_a-y_c}{l}+\frac{y_b-y_c}{l'}\right\}$$

$$-\frac{1}{l}\Sigma_c^a[P(l^2-z^2)z]-\frac{1}{l'}\Sigma_c^b[P'(l'^2-z'^2)z']$$

$$=M_al+2M_c\,(l+l')+M_bl' \quad . \quad (16)$$

Equation (16) then expresses the theorem of three moments for a girder having a constant moment of inertia I, and deflected by weights applied in the span l at distances z from a, and also by weights in the span l' at distances z' from b.

Let us also take the particular case of equation (15) when the moment of inertia is invariable and the piers on a level; then $i=1$, and if we let A_0 and $A_0{}'$ be the positive moment areas due to the weights we have

$$6\left\{\frac{1}{l}A_0\bar{x}_0+\frac{1}{l'}A_0{}'\bar{x}'_0\right\}=$$

$$M_al+2M_c\,(l+l')+M_bl' \quad . \quad . \quad (17)$$

This form of the equation of three moments was first given by Greene.*

The advantage to be derived in discussing this theorem in terms of the bending moments, instead of the applied weights is evident both in the analytical and the graphical treatment. The extreme complexity of the ordinary formulae arises from their being obtained in terms of the weights.

In order to complete the analytic solution of the continuous girder in the general case of equation (15), it is only necessary to use the well known equations,

$$M=M_c+S_cz_0-\Sigma_c^o(Pz_0) \quad . \quad . \quad (18)$$

$$S_c=\frac{1}{l}\,[M_a-M_c+\Sigma_c^a(Pz)] \quad . \quad . \quad (19)$$

$$S_c{}'=\frac{1}{l'}\,[M_b-M_c+\Sigma_c^b\,(Pz')\,] \quad . \quad (20)$$

$$R_c=S_c+S_c{}' \quad . \quad . \quad . \quad . \quad . \quad (21)$$

$$S=S_c-\Sigma_c^o(P) \quad . \quad . \quad . \quad . \quad . \quad (22)$$

In (18) M is the bending moment at any point O in the span l, S_c is the shear at c due to the weights in the span l, and z_0 is the distance from O towards c of the applied forces P and S_c in the segment Oc.

Equation (19) is derived from (18) by taking O at a, and (20) is obtained similarly in the span l'. R_c is the reaction of the pier at c. S is the shear at O in the span l. These equations also complete the solution of the cases treated in (16) and (17).

CHAPTER XII.

THE FLEXIBLE ARCH RIB AND STIFFENING TRUSS.

Whenever the moment of inertia of an arch rib is so small, that it cannot afford a sufficient resistance to hold in equilibrium the bending moments due to the weights, it may be termed a flexible rib.

It must have a sufficient cross section to resist the compression directly along the rib, but needs to be stiffened by a truss, which will most conveniently be made straight and horizontal. The rib may have a large number of hinge joints which must be rigidly connected with the truss, usually by vertical parts. It is then perfectly flexible.

If, however, the rib be continuous without joints, or have blockwork joints, it may nevertheless be treated as if perfectly flexible, as this supposition will be approximately correct and on the side of safety, for the bending moments induced in the truss will be very nearly as great as if the rib were perfectly flexible, in case the same weight would cause a much greater deflection in the rib than in the truss. It will be sufficient to describe the construction for the flexible rib without a figure, as the construction can afford no difficulties after the constructions already given have been mastered.

Lay off on some assumed scale the applied weights as a load line, and let us call this vertical load line ww'. Divide the span into some convenient number of equal parts by verticals, which will divide the curve a of the rib into segments. From some point b as a pole draw a pencil of rays parallel to the segments of a, and across this pencil draw a vertical line uu', at such a distance from b that the distance uu' between the extreme rays of the pencil is equal to ww'. Then the segments of uu' made by the rays of the pencil are the loads which the arch rib would sus-

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* Graphical Method for the Analysis of Bridge Trusses. Chas. E. Greene. Published by D. Van Nostrand. New York, 1875.

tain in virtue of its being an equilibrium polygon, and they would induce no bending moments if applied to the arch. The actual loads in general are differently distributed. By Prop. VI the bending moments induced in the truss are those due to the difference between the weight actually resting on the arch at each point, and the weight of the same total amount distributed as shown by the segments of the line *uu'*.

Now lay off a load line *vv'* made up of weights which are these differences of the segments of *uu'* and *ww'*, taking care to observe the signs of these differences. The algebraic sum of all the weights *vv'* vanishes when the weights which rest on the piers are included, as appears from inspection of the construction in the lower part of Fig. 10. The construction above described will differ from that in Fig. 10 in one particular. The rib will not in general be parabolic, and the loads which it will sustain in virtue of its being an equilibrium polygon will not be uniformly distributed, hence the differences which are found as the loading of the stiffening truss do not generally constitute a uniformly distributed load.

The horizontal thrust of the arch is the distance of *uu'* from *b* measured on the scale on which the loads are laid off, and the thrust along the arch at any point is length of the corresponding ray of the pencil between *b* and *uu'*. These thrusts depend only on the total weight sustained, while the bending moments of the stiffening truss depend on the manner in which it is distributed, and on the shape of the arch.

Having determined thus the weights applied to the stiffening truss, it is to be treated as a straight girder, by methods previously explained according to the way in which it is supported at the piers.

The effect of variations of temperature is to make the crown of the arch rise and fall by an amount which can be readily determined with sufficient exactness, (see Rankine's Applied Mechanics Art. 169). This rise or fall of the arch produces bending moments in the stiffening truss, which is fastened to the tops of the piers, which are the same as would be produced by a positive or negative loading, causing the same deflection at the center and distributed in the same manner as the segments of *uu'*: for it is such a distribution of loads or pressures which the rib can sustain or produce. A similar set of moments can be induced in the stiffening truss by lengthening the posts between the rib and truss.

When this deflection and the value of *EI* in the truss are known, these moments can be at once constructed by methods like those already employed. A judicious amount of cambering of this kind is of great use in giving the structure what may be called "initial stiffness." The St. Louis Arch is wanting in initial stiffness to such an extent that the weight of a single person is sufficient to cause a considerable tremor over an entire span. This would not have been possible had the bridge consisted of an arch stiffened by a truss which was anchored to the piers in such a state of bending tension as to exert considerable pressure upon the arch. This tension of the truss would be relieved to some extent during the passage of a live load.

The arch rib with stiffening truss, is a form of which many wooden bridges were erected in Pennsylvania in the earlier days of American railroad building, but its theory does not seem to have been well understood by all who erected them, as the stiffening truss was itself usually made strong enough to bear the applied weights, and the arch was added for additional security and stiffness, while instead of anchoring the truss to the piers and causing it to exert a pressure on the arch, a far different distribution of pressures was adopted. Quite a number of bridges of this pattern are figured by Haupt* from the designs of the builders, but most of them show by the manner of bracing near the piers that the engineers who designed them did not know how to take advantage of the peculiarities of this combination. This further appears from the fact, that the trussing is not usually continuous.

A good example, however, of this combination constructed on correct principles is very fully described by Haupt on pages 169 *et seq.* of his treatise. It is a wooden bridge over the Susquehanna River, 5½ miles from Harrisburg on the

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* Theory of Bridge Construction. Herman Haupt, A.M. New York. 1853.

Pennsylvania Railroad, and was designed by Haupt. It consists of twenty-three spans of 160 feet each from center to center of piers. The arches have each a span of $149\frac{1}{4}$ feet and a rise of 20 ft. 10 in., and are stiffened by a Howe Truss which is continuous over the piers and fastened to them. It was erected in 1849. Those parts which were protected from the weather have remained intact, while other parts have been replaced, as often as they have decayed, by pieces of the original dimensions. This bridge, though not designed for the heavy traffic of these days, still stands after twenty-eight years of use, a proof of the real value of this kind of combination in bridge building.

CHAPTER XIII.

THE ARCH OF MASONRY.

Arches of stone and brick have joints which are stiff up to a certain limit beyond which they are unstable. The loading and shape of the arch must be so adjusted to each other that this limit shall not be exceeded. This will appear in the course of the ensuing discussion.

Let us take for discussion the brick arch erected by Brunel near Maidenhead England, to serve as a railway viaduct.

It is in the form of an elliptic ring, as represented in Fig. 14, having a span of 128 ft. with a rise of $24\frac{1}{4}$ feet. The thickness of the ring at the crown is $5\frac{1}{4}$ ft., while at the pier the horizontal thickness is 7 ft. 2 inches.

Divide the span into an even number of equal parts of the type bb, and with a radius of half the span describe the semicircle gg. Let $ba=24\frac{1}{4}$ ft. be the rise of the intrados, and from any convenient point on the line bb as b_5 draw lines to a and g. These lines will enable us to find the ordinates ba of the ellipse of the intrados from the ordinates bg of the circle, by decreasing the latter in the ratio of bg to ba. For example, draw a horizontal through g_3 cutting b_5g at i_3, then a vertical through i_3, cutting b_5a at j_3, then will a horizontal through j_3 cut off a_3b_3 the ordinate of the ellipse corresponding to b_3g_3 in the circle, as appears from known properties of the ellipse.

Similarly let $bq=64$ ft. + 7 ft. 2 in., and with bq as radius describe a semicircle. Let $bd=24\frac{1}{4}$ ft. + $5\frac{1}{4}$ ft. be the rise

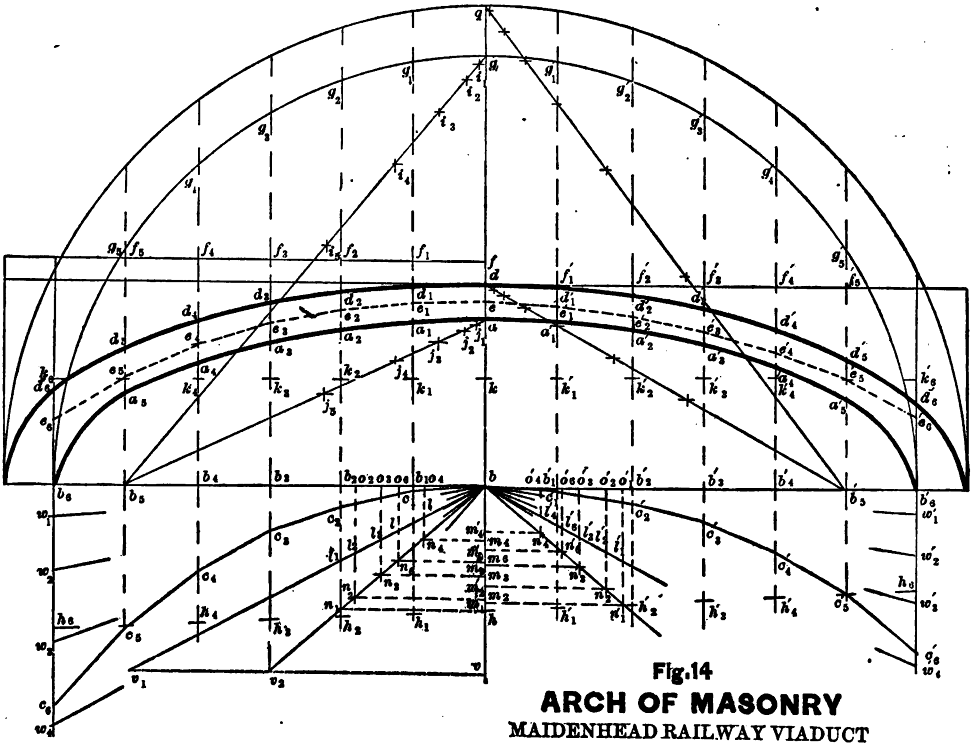


Fig.14
ARCH OF MASONRY
MAIDENHEAD RAILWAY VIADUCT

of the extrados, and from any convenient point on *bb*, as b_6' draw lines to *d* and *q*. These will enable us to find the ordinates *bd* of the ellipse of the extrados, from those of the circle, by decreasing the latter in the ratio of *bq* to *bd*. By this means, as many points as may be desired, can be found upon the intrados and extrados; and these curves may then be drawn with a curved ruler. We can use the arch ring so obtained for our construction, or multiply the ordinates by any convenient number, in case the arch is too flat for convenient work. Indeed we can use the semicircular ring itself if desirable. We shall in this construction employ the arch ring *ad* which has just been obtained.

We shall suppose that the material of the surcharge between the extrados and a horizontal line tangent at *d* causes by its weight a vertical pressure upon the arch. That this assumption is nearly correct in case this part of the masonry is made in the usual manner, cannot well be doubted. Rankine, however, in his Applied Mechanics assumes that the pressures are of an amount and in a direction due to the conjugate stresses of an homogeneous, elastic material, or of a material which like earth has an angle of slope due to internal friction. While this is a correct assumption, in case of the arch of a tunnel sustaining earth, it is incorrect for the case in hand, for the masonry of the surcharge needs only a vertical resistance to support it, and will of itself produce no active thrust, having a horizontal component.

This is further evident from Moseley's principle of least resistance, which is stated and proved by Rankine in the following terms:

"If the forces which balance each other in or upon a given body or structure, be distinguished into two systems, called respectively, *active* and *passive*, which stand to each other in the relation of cause and effect, then will the passive forces be the least which are capable of balancing the active forces, consistently with the physical condition of the body or structure.

For the passive forces being caused by the application of the active forces to the body or structure, will not increase after the active forces have been balanced by them; and will, therefore, not increase beyond the least amount capable of balancing the active forces."

A surcharge of masonry can be sustained by vertical resistance alone, and therefore will exert of itself a pressure in no other direction upon the haunches of the arch. Nevertheless this surcharge will afford a resistance to horizontal pressure if produced by the arch itself. So that when we assume the pressures due to the surcharge to be vertical alone, we are assuming that the arch does not avail itself of one element of stability which may possibly be employed, but which the engineer will hesitate to rely upon, by reason of the inferior character of the masonry usually found in the surcharge. The difficulty is usually avoided, as in that beautiful structure, the London Bridge, by forming a reversed arch over the piers which can exert any needed horizontal pressure upon the haunches. This in effect increases by so much the thickness of the arch ring at and near the piers.

The pressure of earth will be treated in connection with the construction for the Retaining Wall. On combining the pressures there obtained with the weight, the load which a tunnel arch sustains, may be at once found, after which the equilibrium polygon may be drawn and a construction executed, similar in its general features to that about to be employed in the case before us.

Let us assume that the arch is loaded with a live load extending over the left half of the span, and having an intensity which when reduced to masonry of the same specific gravity as that of which the viaduct is built, would add a depth *df* to the surcharge. Now if the number of parts into which the span is divided be considerable, the weights which may be supposed to be concentrated at the points of division vary very approximately as the quantities of the type *af*. This approximation will be found to be sufficiently exact for ordinary cases; but should it be desired to make the construction exact, and also to take account of the effect of the obliquity of the joints in the arch ring, the reader will find the method for obtaining the centers of gravity, and constructing the weights, in Woodbury's Treatise on the Stability of the Arch pp. 405 *et seq.* in which is given Poncelet's graphical solution of the arch.

With any convenient pole distance, as one half the span, lay off the weights. We have used *b* as the pole and made $b_6w_1 = \frac{1}{2}$ the weight at the crown $= \frac{1}{4}(af+ad) = b_6'w_1'$, $w_1w_2 = a_1f_1$, $w_2w_3 = a_2f_2$, etc. Several of the weights near the ends of the span are omitted in the Figure; viz., w_4w_5, etc. From the force polygon so obtained, draw the equilibrium polygon *c* as previously explained.

The equilibrium polygon which expresses the real relations between the loading and the thrust along the arch, is evidently one whose ordinates are proportional to the ordinates of the polygon *c*.

It has been shown by Rankine, Woodbury and others, that for perfect stability, —*i.e*, in case no joint of the arch begins to open, and every joint bears over its entire surface,—that the point of application of the resultant pressure must everywhere fall within the middle third of the arch ring. For if at any joint the pressure reaches the limit zero, at the intrados or extrados, and uniformly increases to the edge farthest from that, the resultant pressure is applied at one third of the depth of the joint from the farther edge.

The locus of this point of application of the resultant pressure has been called the "curve of pressure," and is evidently the equilibrium curve due to the weights and to the actual thrust in the arch. If then it be possible to use such a pole distance, and such a position of the pole, that the equilibrium polygon can be inscribed within the inner third of the thickness of the arch ring, the arch is stable. It may readily occur that this is impossible, but in order to ensure sufficient stability, no distribution of live load should be possible, in which this condition is not fulfilled.

We can assume any three points at will, within this inner third, and cause a projection of the polygon *c* to pass through them, and then determine by inspection whether the entire projection lies within the prescribed limits. In order to so assume the points that a new trial may most likely be unnecessary, we take note of the well known fact, that in arches of this character, the curve of pressure is likely to fall without the prescribed limits near the crown and near the haunches. Let us assume *e* at the middle of the crown, e_5' at the middle of $a_5'd_5'$, and e_5 near the lower limit on a_5d_5. This last is taken near the lower limit, because the curvature of the left half of the polygon is more considerable than the other, and so at some point between it and the crown it may possibly rise to the upper limit. The same consideration would have induced us to raise e_5' to the upper limit, were it not likely that such a procedure would cause the polygon to rise above the upper limit on the right of e_5'.

Draw the closing line *kk* through e_5e_5', and the corresponding closing line *hh* through c_5c_5', and decrease all the ordinates of the type *hc* in the ratio of *hb* to *ke*, by help of the lines *bn* and *bl*, in a manner like that previously explained. For example $h_3c_3=n_3o_3$, and $l_3o_3=k_3e_3$. By this means we obtain the polygon *e* which is found to lie within the required limits. The arch is then stable: but is the polygon *e* the actual curve of pressures? Might not a different assumption respecting the three points through which it is to pass lead to a different polygon, which would also lie within the limits? It certainly might. Which of all the possible curves of pressure fulfilling the required condition, is to be chosen, is determined by Moseley's principle of least resistance, which applied to the case in hand, would oblige us to choose that curve of all those lying within the required limits, which has the least horizontal thrust, *i.e.* the smallest pole distance. It appears necessary to direct particular attention to this, as a recent publication on this subject asserts that the true pressure line is that which approaches nearest to the middle of the arch ring, so that the pressure on the most compressed joint edge is a minimum; a statement at variance with the theorem of least resistance as proved by Rankine.

Now to find the particular curve which has the least pole distance, it is evidently necessary that the curve should have its ordinates as large as possible. This may be accomplished very exactly, thus: above e_1 where the polygon approaches the upper limit more closely than at any other point near the crown, assume a new position of e_1 at the upper limit; and below e_4' where it approaches the lower limit most nearly on the right, assume a new position of e_4' at the lower limit. At the left e_8 may be retained. Now on passing the polygon through these points it will fulfill the second condition, which is imposed by the principle of least resistance.

A more direct method for making the polygon fulfill the required condition will be given in Fig. 18.

It is seen in the case before us, the changes are so minute that it is useless to find this new position of the polygon, and its horizontal thrust. The thrust obtained from the polygon *e* in its present position is sufficiently exact. The horizontal thrust in this case is found from the lines *bn* and *bl*. Since $2vv_2$ is the horizontal thrust, *i.e.* pole distance of the polygon *c*, $2vv_1$ is the horizontal thrust of the polygon *e*.

By using this pole distance and a pole properly placed, we might have drawn the polygon *e* with perhaps greater accuracy than by the process employed, but that being the process employed in Figs. 2, 3, etc., we have given this as an example of another process.

The joints in the arch ring should be approximately perpendicular to the direction of the pressure, *i.e.* normal to the curve of pressures.

With regard to what factor of safety is proper in structures of this kind, all engineers would agree that the material at the most exposed edge should never be subjected to a pressure greater than one fifth of its ultimate strength. Owing to the manner in which the pressure is assumed to be distributed in those joints where the point of application of the resultant is at one third the depth of the joint from the edge, its intensity at this edge is double the average intensity of the pressure over the entire joint. We are then led to the following conclusion, that the total horizontal thrust (or pressure on any joint) when divided by the area of the joint where this pressure is sustained ought to give a quotient at least ten times the ultimate strength of the material. The brick viaduct which we have treated is remarkable in using perhaps the smallest factor of safety in any known structure of this class, having at the most exposed edge a factor of only 3½ instead of 5.

It may be desirable in a case like that under consideration, to discuss the changes occuring during the movement of the live load, and that this may be effected more readily, it is convenient to draw the equilibrium polygons due to the live and dead loads separately. The latter can be drawn once for all, while the former being due to a uniformly distributed load can be obtained with facility for different positions of the load. The polygon can be at once combined into a single polygon by adding the ordinates of the two together. Care must be taken, however, to add together only such as have the same pole distance. In case the construction which has been given should show that the arch is unstable, having no projection of the equilibrium polygon which can be inscribed within the middle third of the arch ring, it is possible either to change the shape of the arch slightly, or increase its thickness, or change the distribution of the loading. The last alternative is usually the best one, for the shape has been chosen from reasons of utility and taste, and the thickness from consideration of the factor of safety. If the center line of the arch ring (or any other line inscribed within the middle third) be considered to be an equilibrium polygon, and from a pole, lines be drawn parallel to the segments of this polygon, a weight line can be found which will represent the loading needed to make the arch stable. If this load line be compared with that previously obtained, it will be readily seen where a slight additional load must be placed, or else a hollow place made in the surcharge, such as will render the arch stable. In general, it may be remarked, that an additional load renders the curvature of the line of pressures sharper under it, while the removal of any load renders the curve straighter under it.

The foregoing construction is unrestricted, and applies to all unsymmetrical forms of arches or of loading, or both. As previously mentioned, a similar construction applies to the case of an arch sustaining the pressure of water or earth; in that case, however, the load is not applied vertically and the weight line becomes a polygon.

CHAPTER XIV.

RETAINING WALLS AND ABUTMENTS.

Let $aa'b'b$ in Fig. 15 represent the cross section of a wall of masonry which retains a bank of earth having a surface aa_6. Assume that the portion of the wall and earth under consideration is bounded by two planes parallel to the plane of the paper, and at a unit's distance from each other: then any plane containing the edge of the wall at b, as ba_0, ba_1, etc., cuts this solid in a longitudinal section, which is a rectangle having a width of one unit, and a length ba_0, ba_1, etc.

The resultant of the total pressure distributed over any one of these rectangles of the type ba is applied at one-third of that distance from b: *i.e.* the resultant pressure exerted by the earth against the rectangle at ba_0 is applied at a distance of $bk = \frac{1}{3}\, ba_0$ from b.

That the resultant is to be applied at this point, is due to the fact that the distributed pressure increases uniformly as we proceed from any point a of the surface toward b: the center of pressure is then at the point stated, as is well known.

Again, the direction of the pressures against any vertical plane, as that at ba_0, is parallel to the surface aa_6. This fact is usually overlooked by those who treat this subject, and some arbitrary assumption is made as to the direction of the pressure.

That the thrust of the earth against a vertical plane is parallel to the ground surface is proved analytically in Rankine's Applied Mechanics on page 127; which proof may be set forth in an elementary manner by considering the small parallelopiped mn, whose upper and lower surfaces are parallel to the ground surface. Since the pressure on any plane parallel to the surface of the ground is due to the weight of the earth above it, the pressure on such a plane is vertical and uniformly distributed. If mn were a rigid body, it would be held in equilibrium by these vertical pressures, which are, therefore, a system of forces

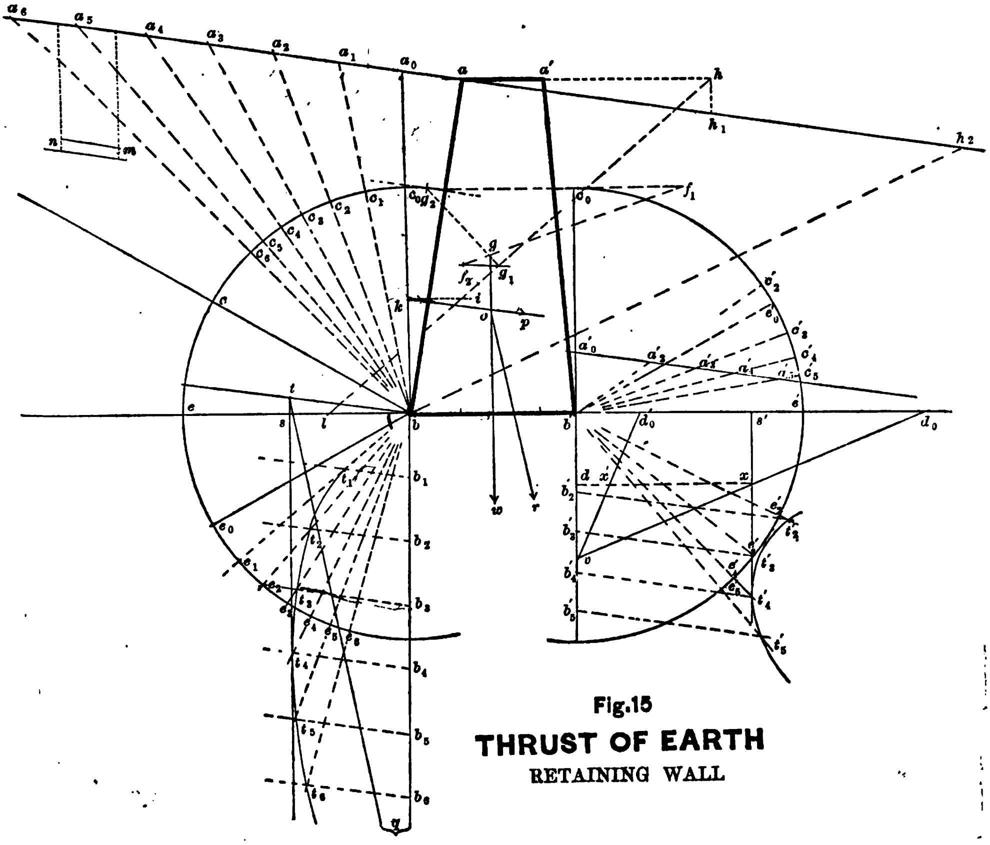


Fig.15

THRUST OF EARTH

RETAINING WALL

in equilibrium; but as mn is not rigid it must be confined by pressures distributed over each end surface, which last are distributed in the same manner on each end, because each is at the same depth below the surface. Now the vertical pressures and end pressures hold mn in equilibrium: they therefore form a system in equilibrium. But the vertical pressures are independently in equilibrium, therefore the end pressures alone form a system which is independently in equilibrium. That this may occur, and no couple be introduced, these must directly oppose each other; *i.e.* be parallel to the ground line aa_4.

Draw $kp \parallel aa_4$, it then represents the position and direction of the resultant pressure upon the vertical ba_0. Draw the horizontal ki, then is the angle ikp called the *obliquity* of the pressure, it being the angle between the direction of the pressure and the normal to the plane upon which the pressure acts.

Let $ebc = \Phi$ be the *angle of friction*, *i.e.* the inclination which the surface of ground would assume if the wall were removed.

The obliquity of the pressure exerted by the earth against any assumed plane, such as ba_3 or ba_4, must not exceed the angle of friction; for should a greater obliquity occur the prism of earth, a_0ba_3 or a_0ba_4, would slide down the plane, ba_3 or ba_4, on which such obliquity is found.

For dry earth Φ is usually about 30°; for moist earth and especially moist clay, Φ may be as small as 15°. The inclination of the ground surface aa_4 cannot be greater than Φ.

Now let the points a_1, a_2, a_3, etc., be assumed at any convenient distances along the surface: for convenience we have taken them at equal distances, but this is not essential. With b as a center and any convenient radius, as bc, describe a semi-circumference cutting the lines ba_1, ba_2, etc. at c_1, c_2, etc. Make $ee_0 = ec$; also $e_0e_1 = c_0c_1$, $e_0e_2 = c_0c_2$, etc.: then be_0 has an obliquity Φ with ba_0, as has also be_1 with ba_1, be_2 with ba_2, etc.; for $a_0be_0 = a_1be_1 = a_2be_2 = 90° + \Phi$.

Lay off bb_1, bb_2, bb_3, etc., proportional to the weights of the prisms of earth a_0ba_1, a_0ba_2, a_0ba_3, etc.: we have effected this most easily by making $a_0a_1 = bb_1$, $a_0a_2 = bb_2$, $a_0a_3 = bb_3$, etc. Through b, b_1 b_2, etc., draw parallels to kp; these will intersect be_0, be_1, be_2, etc., at b, t_1, t_2, etc. Then is bb_1t_1 the triangle of forces holding the prism a_0ba_1 in equilibrium, just as it is about to slide down the plane ba_1, for bb_1 represents the weight of the prism, b_1t_1 is the known direction of the thrust against ba_0, and bt_1 is the direction of the thrust against ba_1 when it is just on the point of sliding: then is t_1b_1 the greatest pressure which the prism can exert against ba_0. Similarly t_2b_2 is the greatest pressure which the prism a_0ba_2 can exert. Now draw the curve $t_1t_2t_3$, etc., and a vertical tangent intersecting the parallel to the surface through b at t; then is tb the greatest pressure which the earth can exert against ba_0. This greatest pressure is exerted approximately by the prism or wedge of earth cut off by the plane ba_4, for the pressure which it exerts against the vertical plane through b is almost exactly $b_4t_4 = bt$. This is Coulomb's "wedge of maximum thrust" correctly obtained: previous determinations of it have been erroneous when the ground surface was not level, for in that case the direction of the pressure has not been ordinarily assumed to be parallel to the ground surface.

In case the ground surface is level the wedge of maximum thrust will always be cut off by a plane bisecting the angle cbc_0, as may be shown analytically, which fact will simplify the construction of that case, and enable us to dispense with drawing the thrust curve tt.

The pressure tb is to be applied at k, and may tend either to overturn the wall or to cause it to slide.

In order to discuss the stability of the wall under this pressure, let us find the weight of the wall and of the prism of earth aba_0. Let us assume that the specific gravity of the masonry composing the wall is twice that of earth. Make $a'h = bb'$, then the area $abb'a' = abh = abh_1$; and if $ah_2 = 2ah$, then ah_2 represents the weight of the wall reduced to the same scale as the prisms of earth before used. Since aa_0 is the weight of aba_0, a_0h_2 is the weight of the mass on the right of the vertical ba_0 against which the pressure is exerted.

Make $bq = a_0h_2$, and draw tq, which then represents the direction and amount of the resultant to be applied at o where the resultant pressure applied at k intersects the vertical gw through the center of gravity g of the mass $aa_0bb'a'$. The center of gravity g is constructed in the following manner. Lay off $a'h=bb'$, and $bl=aa'$; and join hl. Join also the middle points of ab and $a'b'$: the line so drawn intersects hl at g_1 the center of gravity of $aa'b'b$. Find also the center of gravity g_2, of aba_0, which lies at the intersection of a line parallel to aa_0, and cutting ba_0 at a distance of $\frac{1}{3}\,ba_0$ from a_0 and of a line from b bisecting aa_0. Through g_2 and g_1 draw parallels, and lay off g_2f_1 and g_1f_2 on them proportional to the weights applied at g_1 and g_2 respectively. We have found it convenient to make $g_2f_1=\frac{1}{2}ah_2$, and $g_1f_2=\frac{1}{2}\,aa_0$. Then f_1f_2 divides g_1g_2 inversely as the applied weights; and g, the point of intersection, is the required center of gravity.

Let *or* be parallel to *tq*; since it intersects bb' so far within the base, the wall has sufficient stability against overturning. The base of the wall is so much greater than is necessary for the support of the weight resting upon it, that engineers have not found it necessary that the resultant pressure should intersect the base within the middle third of the joint. The practice of English engineers, as stated by Rankine, is to permit this intersection to approach as near b' as $\frac{1}{8}bb'$, while French engineers permit it to approach as near as $\frac{1}{5}bb'$ only. In all cases of buttresses, piers, chimneys, or other structures which call into play some fraction of the ultimate strength of the material, or ultimate resistance of the foundation as great as one tenth, or one fifteenth, the point should not approach b' nearer than $\frac{1}{3}\,bb'$.

Again, let the angle of friction between the wall and the earth under it be Φ': then in order that the thrust at k may not cause the wall to slide, the angle *wor* must be less than Φ'.

When, however, the angle Φ' is less than *wor* it becomes necessary to gain additional stability by some means, as for example by continuing the wall below the surface of the ground lying in front of it. Let $a_0'a_8'$ be the surface of the ground which is to afford a *passive* resistance to the thrust of the wall: then in a manner precisely analogous to that just employed for finding the greatest active pressure which earth can exert against a vertical plane, we now find the least passive pressure which the earth in front of the wall will sustain without sliding up some plane such as $b'a_3'$ or $b'a_4'$, etc. The difference in the two cases is that in the former case friction hindered the earth from sliding down, while it now hinders it from sliding up the plane on which it rests.

Lay off $e'e_0'=ee_0$; then taking any points $a_2'a_3'$, etc. on the ground surface, make $e_0'e_2'=c_0'c_2'$, $e_0'e_3'=c_0'c_3'$, etc.

Lay off $b'b_2'=a_0'a_2'$, etc., and drawing parallels through b_2', b_3', etc., we obtain the thrust curve $t_2't_3'$, etc.

The small prism of earth between $b'a_0'$ and the wall adds to the stability of the wall, and can be made to enter the construction if desired, in the same manner as did aba_0.

The vertical tangent through s' shows us that the earth in front of the wall can withstand a thrust having a horizontal component $b's'$ measured on a scale such that $b'b_2'=a_0'a_2'$ is the weight of the prism of earth $a_0'b'a_2'$.

This scale is different from that used on the left. To reduce them to the same scale lay off from b', the distances $b'd_0$ and $b'd_0'$ proportional to the perpendiculars from b on aa_4 and b' on $a_1'a_4'$ respectively. In the case before us, as the ground surfaces are parallel, we have made $b'd_0=ba_0$ and $b'd_0'=b'a_8'$.

Then from any convenient point on $b'b_4'$, as v, draw vd_0 and vd_0': these lines will reduce from one scale to the other. We find then that $x'd$ is the thrust on the scale at the left corresponding to $xd=b's'$ on the right: *i.e.*, the earth under the surface assumed at the right can withstand something over one fourth of the thrust sb at the left.

It will be found that a certain small portion of the earth near a_0' has a thrust curve on the left of b', but as it is not needed in our solution it is omitted.

If any pressure is required in pounds, as for example sb, it is founds as follows:—the length of ah_2 is to that of sb as the weight of $bb'aa'$ in lbs. is to the pressure sb in lbs.

Frequently the ground surface is not a plane, and when this is the case it often consists of two planes as ad, da_7 Fig. 16. In that case, draw some convenient line as ad_7, and lay off ad_1, d_1d_2, etc. at will, which for convenience we have made equal. Draw d_1a_1, d_2a_2, etc. parallel to bd, and join ba_1, ba_2, etc.: then are the

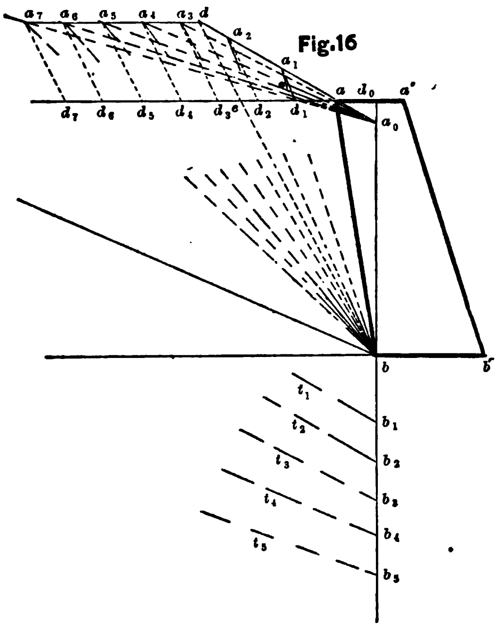


Fig. 16

triangles bda, bda_1, bda_2, bda_3, etc. proportional in area to the lines ea, ea_1, etc. Hence the weights of the prisms of earth baa_1, baa_2, etc., are proportional to ad_1, ad_2, etc.

In case ab slopes backward the part of the wall at the left of the vertical ba_0 rests upon the earth below it sufficiently to produce the same pressure which would be produced if baa_0 were a prism of earth. The weights of the wedges which produce pressures, and which are to be laid off below b, are then proportional to $d_0d_1=bb_1$, $d_0d_2=bb_2$, etc. The direction of the pressures of the prisms at the right of bd are parallel to ad; but upon taking a larger prism the direction may be assumed to be parallel to a_0a_3, a_0a_4, etc., which is very approximately correct. Now draw $b_1t_1 \parallel a_0a_1$, $b_4t_4 \parallel a_0a_4$, etc.; and complete the construction for pressure precisely as in Fig. 15, using for resultant pressure the direction and amount of that due to the wedge of maximum pressure thus obtained.

In finding the stability of the wall, it will be necessary to find the weight and center of gravity of the wall itself, minus a prism of earth baa_0, instead of plus this prism as in Fig. 15; for it is now sustained by the earth back of the wall.

When the back of the wall has any other form than that above treated, the vertical plane against which the pressure is determined should still pass through the lower back edge of the wall.

In case the wall is found to be likely to slide upon its foundations when these are level, a sloping foundation is frequently employed, such that it shall be nearly perpendicular to the resultant pressure upon the base of the wall. The construction employed in Fig. 15 applies equally to this case.

The investigation of the stability of any abutment, buttress, or pier, against overturning and against sliding, is the same as that of the retaining wall in Fig. 15. As soon as the amount, direction, and point of application, of the pressure exerted against such a structure is determined, it is to be treated precisely as was the resultant pressure kp in Fig. 15.

In the case of a reservoir wall or dam, the construction is simplified from the fact that, since the surface of water is level and the angle of friction vanishes, the resultant pressure is perpendicular to the surface upon which the water presses. It is useful to examine this as a case of our previous construction. In Fig. 17, let abb' be the cross-section of the dam; then the wedge of maximum pressure against ba_0 is cut off by the plane ba_1 when $cba_1=45°$, *i.e.* ba_1 bisects cba_0 as before stated.

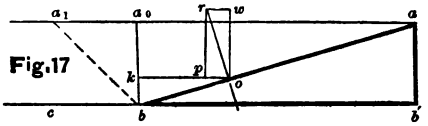


Fig.17

This produces a horizontal resultant pressure at k equal to the weight of the wedge. Now the total pressure on ab is the resultant of this pressure, and the weight of the wedge aba_0. The forces to be compounded are then proportional to the lines $a_1a_0=bv_0$ and aa_0. By similarity of triangles it is seen that ro the resultant is perpendicular to ab.

It is seen that by making the inclination of ab small, the direction of ro can be made so nearly vertical that the dam will be retained in place by the pressure of the water alone, even though the dam be a wooden frame, whose weight may be disregarded.

We can now construct the actual pressures to which the arch of a tunnel surcharged with water or earth is subjected. Suppose, for example, we wish to find the pressure of such a surcharge on the voussoir $a_4d_4d_5a_5$ Fig. 14. Find the resultant pressure against a vertical plane extending from d_5 to the upper surface of the surface and call it p_5. Draw a horizontal through d_4 and let its intersection with the vertical just mentioned he called d_5''. Find the resultant pressure against the vertical plane extending from d_5'' to the surface, and call it p_5'. Now let $p_5''=p_5-p_5'$ and let it be applied at such a point of d_5d_5'' that p_5 shall be the resultant of p_5' and p_5''. Then will the resultant pressure against the voussoir be the resultant of p_5'' and the weight of that part of the surcharge directly above it.

FOUNDATIONS IN EARTH.

A method similar to that employed in the determination of the pressure of earth against a retaining wall, or a tunnel arch, enables us to investigate the stability of the foundations of a wall standing in earth.

Suppose in Fig. 15 that the wall $abb'a'$ is a foundation wall, and that the pressure which it exerts upon the plane bb' is vertical, being due to its own weight and the weight of the building or other load which it sustains. Now consider a vertical plane of one unit in height, say, as bb_1; and determine the resultant pressure against it on the supposition that the pressure is produced by a depth of earth at the right of it, sufficient to produce the same vertical pressure on bb' which the wall and its load do actually produce. In other words we suppose the wall and load replaced by a bank of earth having its upper surface horizontal and weighing the same as the wall and load. Call the upper surface z, and find the pressure against the vertical plane zb due to the earth under the given level surface; similarly, find the pressure against zb_1. The surface being level, the maximum pressure, as previously stated will be due to a wedge cut off by a plane bisecting the angle between bz and a plane drawn from b at the inclinatian Φ, of the limiting angle of friction. This enables us to find the horizontal pressures against zb and zb_1 directly: their difference is the resultant active pressure against bb_1.

Next, it must be determined what passive pressure the earth at the left of bb_1 can support. The passive resistance of the earth under the surface a against the plane ab as well as that against the plane ab_1 can be found exactly as that was previously found under the surface a'. The difference of these resistances is the resistance which it is possible for bb_1 to support. Indeed bb_1 could support this pressure and afford this resistance even if the active pressure against ab were, at the limit of its resistance, which it is not. The limiting resistance which is thus obtained, is then so far within the limits of stability, that ordinarily, no further factor of safety is needed, and the stability of the foundation is secured, if the active pressure against bb_1 does not exceed the passive resistance. This construction should be made on the basis of the smallest angle of friction Φ which the earth assumes when wet; that being smaller than for dry earth, and hence giving a greater active pressure at the right, and a less resistance at the left.

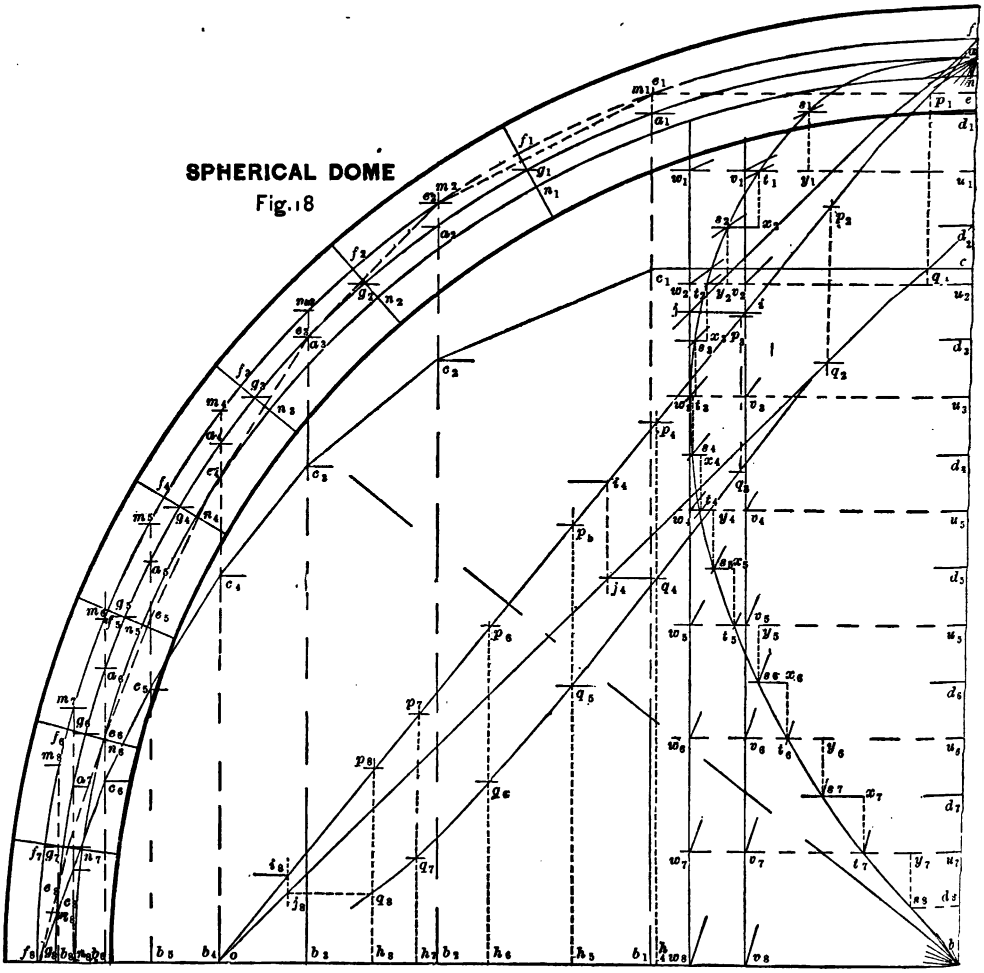
CHAPTER XV.

SPHERICAL DOME OF METAL.

The dome which will be treated in the following construction is hemispherical in shape; but the proposed construction applies equally to domes of any different form generated by the revolution of the arc of some curve about a vertical axis: such forms are elliptic, parabolic or hyperbolic domes, as well as pointed or gothic domes, etc. Let the quadrant *aa* in Fig. 18, represent the part of the meridian section of a thin metallic dome between the crown and the springing circle. The metallic dome is supposed to be so thin that its thickness need not be represented in the Figure: the thickness of a dome of masonry, however, is a matter of prime importance and will be treated subsequently.

In a thin metallic dome the only thrust along a meridian section is necessarily in a direction tangent to that section at each point of it. This consideration will enable us to determine this thrust as well as the hoop tension or compression along any of the conical rings into which the dome may be supposed to be divided by a series of horizontal planes.

Let the height *ab* of the dome be divided into any number of parts, which we have in this case, for convenience, made equal. Let these equal parts of the type du be the distances between horizontal planes such that the planes through the points d_1, d_2, etc., cut small circles from the hemisphere which pass through the point a_1, a_2, etc., and similarly the planes through u_1, u_2, etc., cut small circles which pass through g_1, g_2, etc. Now suppose the thickness of this dome to be uniform, and if *ab* be taken to represent the weight of a quadrantal lune of the dome included between two meridian planes making some small angle with each other; then



SPHERICAL DOME
Fig. 18

from the well-known expression for the area of the zone of a sphere it appears that ad_1 will represent the weight of that part of the lune above a_1d_1. Similarly au_1 is the weight of the lune ag_1; ad_2 the weight of aa_2, etc.

This method of obtaining the weight applies of course in case the dome is any segment of a sphere less than a hemisphere and of uniform thickness. If the thickness increases from the crown, the weights of the zones cut by equi-distant horizontal planes increase directly as the thickness. In case the dome is not spherical the weights must be determined by some process suited to the form of the dome and its variation in thickness.

Now the weight of the lune aa_1 is sustained by a horizontal thrust which is the resultant of the horizontal pressures in the meridian planes by which it is bounded, and by a thrust, as before remarked, in the direction of the tangent at a. Draw a horizontal line through d_1, and through a a parallel to the tangent at a: these intersect at s_1, then is ad_1s_1 the triangle of forces which hold in equilibrium the lune aa_1. Similarly, au_1t_1 is the triangle of forces holding the lune ag_1 in equilibrium, etc. Draw a curve st through the points thus determined. This curve is a well-known cubic which when referred to ba as the axis of x and bg_3 as that of y has for its equation

$$\frac{y^2}{x^2}=\frac{r-x}{r+x}$$

On being traced at the right of a it has in the other quadrant of the dome a part like that here drawn forming a loop; it passes through b at an inclination of 45° and the two branches below b finally become tangent to a horizontal line drawn tangent to the circle aa of the dome. The curve has this remarkable property:—If any line be drawn from a, cutting the curve here drawn and, also, the part below bg_3, the product of these two radii vectores of the curve from the pole a is constant, and the locus of the intersection of the normals at these two points is a parabola.

Draw a vertical tangent to this curve: the point of contact is very near t_3, and g_3, the corresponding point of the dome is almost 52° from the crown a. A determination of this maximum point by means of the equation gives the height of it above b as $\frac{1}{2}(\sqrt{5}-1)\,r$, corresponding to about $51°49'$. Now consider any zone, as, for example, that whose meridian section is g_1a_2: the upper edge is subjected to a thrust whose radial horizontal component is proportional to u_1t_1, while the horizontal thrust against its lower edge is proportional to d_2s_2, and the difference s_2x_2 between these radial forces produces a hoop compression around the zone proportional to s_2x_2. It will be seen that these differences which are of the type sx or ty, change sign at t_3. Hence all parts of the dome above $51°\ 49'$ from the crown, are subjected to a hoop compression which vanishes at that distance from a, while all parts of the dome below this are subjected to hoop tension. This may be stated by saying that a thin dome of masonry would be stable under hoop compression as far as $51°\ 49'$ from the crown, but unstable below that, being liable to crack open along its meridian sections. A thick dome of masonry, however, does not have the resultant thrust at every point of its meridian section in a direction which is tangential to its surface,—this will be discussed later.

It is necessary to determine the actual hoop tension or compression in any ring in order to determine the thickness of the dome such that the metal may not be subjected to too severe a stress.

The rule for obtaining hoop tension (we shall use the word tension to include both tension and compression) is: Multiply the intensity of the radial pressure by the radius of the hoop, the product is the tension at any meridian section of the hoop. The correctness of this rule appears at once from consideration of fluid pressure in a tube, in which it is seen that the tensions at the two extremities of a diameter prevent the total pressure on that diameter from tearing the tube asunder.

Now in the case before us t_1y_1 is the radial force distributed along a certain lune. The number of degrees of which the lune consists is at present undetermined: let it be determined on the supposition that it shall be such a number of degrees as to cause that the total radial force against it shall be equal to the hoop tension. Call the total radial force P and the hoop tension T, then the lune is to be such that $P=T$. Also let θ be the number of degrees in the lune, then $90°\div\theta$ is the number of lunes in a quarter of the dome, and $90\,P\div\theta$ is the radial force against a quarter of the dome, which last must be divided by $\frac{1}{2}\pi$ to obtain the hoop tension; because if p is the intensity of radial pressure, $\frac{1}{2}\pi rp$ is the total pressure against a quadrant and rp, as previously stated, is the hoop tension. The ratio of these is $\frac{1}{2}\pi$, and by this we must divide the total radial pressure in every case to obtain hoop tension`

$$\therefore \frac{180\,P}{\theta\pi}=T, \quad \therefore\ \theta=\frac{180°}{\pi}$$

$$\text{for } P=T \quad \therefore\ \theta=57°.3-$$

This is the number of degrees of which the lune must consist in order that when ab represents its weight, t_1y_1 shall represent the hoop tension in the meridian section a_1g_1. The expression we have found is independent of the radius of the ring, and hence holds for any other ring as g_1a_2, in which s_2x_2 is the hoop tension, etc. To find what fraction this lune is of the whole dome, divide θ by 360°

$$\therefore \frac{\theta}{360}=\frac{180}{360\pi}=\frac{1}{2\pi}=\frac{4}{25}\text{nearly,}$$

from which the scale of weight is easily found, thus; let W be the total weight of the dome and r its radius, then

$2\pi r : W :: 1 : n$, the weight per unit, or the hoop tension per unit of the distances ty or sx.

Distances at or as, on the same scale, represent the thrust tangential to the dome in the direction of the meridian sections, and uniformly distributed over an arc of 57°.3— : *e.g.* if we divide at_2 measured as a force by $\theta\times u_2g_2$ measured as a distance we shall obtain the intensity of the meridian compression at the joint cut from the dome by the horizontal plane through a_2.

Analogous constructions hold for domes not spherical and not of uniform thickness. Approximate results may be obtained by assuming a spherical dome, or a series of spherical zones approximating in shape to the form which it is desired to treat.

CHAPTER XVI.

SPHERICAL DOME OF MASONRY.

Let the dome treated be that in Fig. 18 in which the uniform thickness of the masonry is one-sixteenth of the internal diameter or one-eighth of the radius of the intrados. Divide ab the radius of the center line into any convenient number of equal parts, say eight, at u_1, u_2, etc.: a much larger number would be preferable in actual construction. At the points a_1, a_2, etc., on the same levels with u_1, u_2, etc. pass conical joints normal to the dome, so that b is the vertex of each of the cones.

If we consider a lune between meridian planes making a small angle with each other, the center of gravity of the parts of the lune between the conical joints lie at g_1, g_2, etc. on the horizontal midway between the previous horizontals. These points are not exactly upon the central line aa, but if the number of horizontals is large, the difference is inappreciable. We assume them upon aa. That they fall upon the horizontals through d_1, d_2, etc., midway between those through u_1, u_2, etc., is a consequence of the equality in area between spherical zones of the same height.

In finding the volume of a sphere it may be considered that we take the sum of a series of elementary cones whose bases form the surface of the sphere, and whose height is the radius. Hence, if any equal portions of the surface of a sphere be taken and sectorial solids be formed on them as bases and having their vertices at the center, then the sectorial solids have equal volumes. The lunes of which we treat are equal fractions of such equal solids.

Draw the verticals of the type bg through the centers of gravity g_1, g_2, etc. The weights applied at these points are equal and may be represented by au_1, $u_1u_2=w_1w_2$, etc. Use a as the pole and w_1w_8 as the weight line; and, beginning at the point f_8, draw the equilibrium polygon c due to the weights.

We have used for pole distance the greatest horizontal thrust which it is possible for any segment of the dome to exert upon the part below it, when the hoop compression extends to 51° 49′ from the crown.

Below the point where the compression

vanishes we shall not assume that the bond of the masonry is such that it can resist the hoop tension which is developed. The upper part of the dome will be then carried by the parts of the lunes below this point by their united action as a series of masonry arches standing side by side.

Now it is seen that the curve of equilibrium c, drawn with this assumed horizontal thrust falls within the curve of the lune, which signifies that the dome will not exert so great a thrust as that assumed. By the principle of least resistance, no greater horizontal thrust will be called into action than is necessary to cause the dome to stand, if stability is possible. If a less thrust than that just employed be all that is developed in the dome, then the point where the hoop compression vanishes is not so far as 51° 49′ from the crown, and a longer portion of the lune acts as an arch, than has been supposed by previous writers on this subject,* none of whom, so far as known, have given a correct process for the solution of the problem, although the results arrived at have been somewhat approximately correct.

To ensure stability, the equilibrium curve must be inscribed within the inner third of that part of the meridian section of the lune which is to act as an arch; as appears from the same reasons which were stated in connection with arches of masonry.

And, further, the hoop compression will vanish at that level of the dome where the equilibrium curve, in departing from the crown, first becomes more nearly vertical than the tangent of the meridian section; for above that point the greatest thrust that the dome can exert, cannot be so great as at this point where the thrust of the arch-lune is equal to that of the dome.

Now to determine in what ratio the ordinates of the curve c must be elongated to give those of the curve e which fulfills the required conditions, we draw the line fo, and cut it at p_1, p_2, etc. by the horizontals m_1p_1, m_2p_2, etc., the quantities mb being the ordinates of exterior of the inner third. Again draw verticals through p_1, p_2, etc., and cut them at q_1, q_2, q_3, etc. by horizontals through c_1, c_2, c_3, etc. Through these points draw the curve qq, whose ordinates are of the type qh. Some one of these ordinates is to be elongated to its corresponding ph, and in such a manner that no qh shall then become longer than its corresponding ph. To effect this, draw oq_2 tangent to the curve qq; then will oq_2 enable us to effect the required elongation: *e.g.* let the horizontal through c_4 cut oq_2 at j_4, and then the vertical through j_4 cuts fo at i_4, then is e_4 (which is on the same level with i_4) the new position of c_4. Similarly, we may find the remaining points of the curve e; but it is better to determine the new pole distance, and use this method as a test only.

The curve qq made use of in this construction for finding the ratio lines for so elongating the ordinates of the curve c, that the new ordinates shall be those of a curve e tangent to the exterior line of the inner third, may be applied with equal facility to the construction for the arch of masonry. This furnishes us with a direct method in place of the tentative one employed in connection with Fig. 14.

To find the new pole distance, draw $fj \parallel oq_2$ cutting ww at j, then will i the intersection of the horizontal through j, be the new position of the weight line vv, having its pole distance from a diminished in the required ratio.

The equilibrium curve e will be parallel to the curve of the dome at the points where the new weight line vv cuts the curve st. It should be noticed that the pole distance which we have now determined is still a little too large because the polygon e is circumscribed about the true equilibrium curve; and as the polygon has an angle in the limiting curve mm the equilibrium curve is not yet high enough to be tangent to the limiting curve. If the number of divisions had originally been larger (which the size of our Figure did not permit) this matter would be rectified.

The polygon e is seen at e_3 to fall just without the required limits, this would be partly rectified by slightly decreasing the pole distance as just suggested; the point, however, would still remain just without the limit after the pole distance is decreased, and by so much is the dome unstable. A dome of which the thickness is one fifteenth of the internal diameter, is almost exactly stable.

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* See a paper read before the Royal Inst. of British Architects, "on the Mathematical Theory of Domes," Feb. 6th, 1871. By Edmund Beckett Denison, L.L.D., Q.C., F.R.A.S.

It is a remarkable fact that a semi-cylindrical arch of uniform thickness and without surcharge must be almost exactly three times as thick, viz., the thickness must be about one fifth the span in order that it may be possible to inscribe the equilibrium curve within the inner third.

The only large hemispherical dome, of which I have the dimensions, which is thick enough to be perfectly stable without extraneous aid such as hoops or ties, is the Gol Goomuz at Beejapore, India. It has an internal diameter of 137½ feet, and a thickness of 10 feet, it being slightly thicker than necessary, but it probably carries a load upon the crown which requires the additional thickness.

The hemispherical dome of uniform thickness is a very faulty arrangement of material. It is only necessary to make the dome so light and thin for 51° 49′ from the crown that it cannot exert so great a horizontal thrust as do the thicker lunes below, to take complete advantage of the real strength of this form of structure. A dome whose thickness gradually decreases toward the crown takes a partial advantage of this, but nothing short of a quite sudden change near this point appears to be completely effective.

The necessary thickness to withstand the hoop compression and the meridian thrust can be found as previously shown in the dome of metal.

Domes are usually crowned with a lantern or pinnacle, whose weight must be first laid off below the pole *a* after having been reduced to the same unit as that of the zones of the dome.

Likewise when there is an eye, at the crown or below, the weight of the material necessary to fill the eye must be subtracted, so that *a* is then to be placed below its present position. The construction is then to be completed in the same manner as in Fig, 18.

It is at once seen that the effect of an additional weight, as of a lantern, at the crown, since it moves the point *a* upward a certain distance, will be to cause the curve *st* to have all its points except *b* to the left of their present position, and especially the points in the upper part of the curve, thus making the point of no hoop tension much nearer the crown than in the metallic dome. It will be noticed that the addition of very small weight at the crown will cause the point m_2 of no hoop tension in the dome of masonry to approach almost to the crown, so that then the lunes will act entirely as stone arches with the exception of a very small segment at the crown.

On the contrary, the removal of a segment at the crown, or the decrease of the thickness, or any device for making the upper part of the dome lighter will remove the point of no hoop tension further from the crown, both for the dome of metal and of masonry. In any dome of masonry the thickness above the point of no hoop tension, as determined by the curve *st*, need be only such as to withstand the two compressions to which it is subjected, viz; hoop compression and meridian compression: while below that the lunes acting as arches must be thick enough to cause a horizontal thrust equal to the maximum radial thrust of the dome above the point of no hoop tension.

Several large domes are constructed of more than one shell, to give increased security to the tall lanterns surmounting them: St. Peter's, at Rome, is double, and the Pantheon, at Paris, is triple. The different shells should all spring from the same thick zone below the point of no hoop tension; and the lunes of this thick zone should be able to afford a horizontal thrust equal to the sum of the radial thrusts of all the shells standing upon it.

Attention to this will secure the stability in itself of any dome of masonry spherical or otherwise; and, though I here offer no proof of the assertion, I am led to believe that this is the solution of the problem of constructing the dome of a minimum weight of material, on the supposition that the meridian joints can afford no resistance to hoop tension.

Now, in fact, it is a common device to ensure the stability of large domes by encircling them with iron hoops or chains, or by embedding ties in the masonry; and this case appears to be of sufficient importance to demand our attention.

If the hoop encircles the dome at 51° 49′ or any other less distance from the crown the dome will be a true dome at all points above the hoop. Suppose the hoop to be at 51° 49′, then the curve e should, below that point, be made to pass through the points f_3 and f_8, from which it is seen that the dome may be made thinner than at present, and the horizontal thrust caused will be less. The tension of the hoop would be that due to a radial thrust which is the difference between that given by the curve st for this point and the horizontal thrust (pole distance) of the polygon e when it passes through f_3 and f_8. That the curve e passes through these last mentioned points is a consequence of the principle of least resistance.

Again, suppose another hoop encircles the dome at f_5; the curve e must pass through f_8 and f_5, and in this part of the lune will have a corresponding horizontal thrust. The curve e must also pass through f_5 and f_3, but in this part of the lune will have a horizontal thrust corresponding to it, differing from that in the part between f_8 and f_5: indeed the horizontal thrust in the segment of a dome above any hoop depends exclusively upon that segment and and is unaffected by the zone below the hoop. The tension sustained by the hoop is, however, due to the radial force, which is the difference of the horizontal thrusts of the zones above and below the hoop.

It is seen that the introduction of a second hoop will still further diminish the thickness of lune necessary to sustain the dome, unless indeed the thickness is required to sustain the meridian compression.

Had a single hoop been introduced at f_5 with none above that point, the dome above f_5 should then be investigated, just as if the springing circle was situated at that point. The curve e must then start from f_5, as it before did from f_8, and be made to become tangent to the limiting curve at some point between f_5 and the crown.

By the method here employed for finding the tension of a hoop it is possible to discuss at once the stresses induced in the important modern domes constructed with rings and ribs of metal and having the intermediate panels closed with glass.

On introducing a large number of rings at small distances from each other, it will be seen that the discussion just given leads to the method previously given for the dome of metal.

The dome of St. Paul's, London, is one which has excited much adverse criticism by reason of the novel means employed to overcome the difficulties inherent in so large a dome at so great a height above the foundations of the building. The exterior dome consists of a framework of oak sustained by conical dome of brick which forms the core. There is also a parabolic brick dome under the cone which forms no essential part of the system. Since the conical dome in general presents some peculiarities worthy of notice we will give an investigation of that form of structure as our concluding construction.

CHAPTER XVII.

CONICAL DOME OF METAL.

In Fig. 19, let bd be the axis of the frustum of a metallic cone cut by a vertical plane in the meridian section a. The cone is supposed to have a uniform thickness too small to be regarded in comparison with its other dimensions. Suppose the frustum to be cut by a series of equi-distant horizontal planes as at g_1, g_2, etc., into a series of frustra or rings: then the weight of each ring is proportional to its convex surface. The convex surface of any ring $=2\pi r \times$ slant height; when r is half the sum of the radii of the two bases, *i.e.*, r is the mean radius. Consequently, the weights of these rings, or any given fraction of them included between two meridian planes, is proportional to their mean radii. Let us draw these mean radii d_1a_1, d_2a_2, etc., between the horizontals through g_1,g_2, etc., and use some convenient fraction, say $\frac{1}{3}$, of these quantities of the type da as the weights. The line ii cuts off $\frac{1}{3}$ of each of these: then lay off $du_1=d_1i_1$ as the weight of the ring ag_1, lay off $u_1u_2=d_2i_2$, $u_2u_3=d_3i_3$, etc., as the weights of the rings g_1g_2, g_2g_3, etc.

Draw the line $dt \parallel aa$, it corresponds to the curve st of Fig. 18; then the quantities of the type tu represent the horizontal radial thrust which the cone exerts upon the part below it, while the radial thrust borne by any ring is the difference between two successive quantities of the type tu, *i.e.*, the radial thrust in the ring g_7g_8 is represented by t_8y_8,

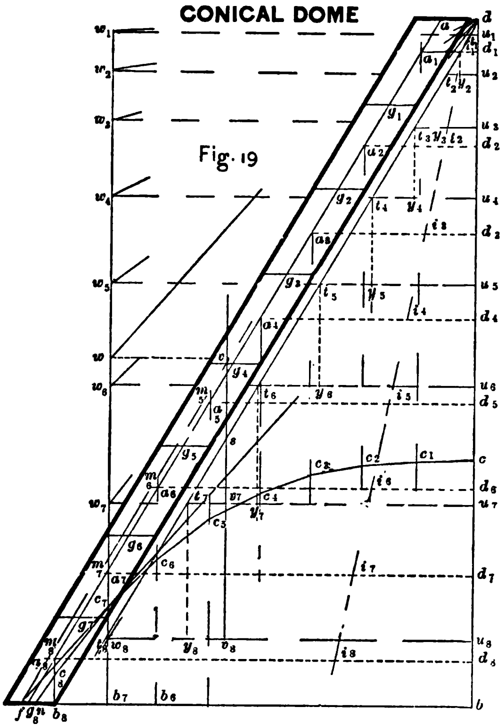

Fig. 19

that in g_6g_7 by t_7y_7, etc. As previously shown in connection with the spherical dome, if the scale of weights be such that du_8 represents a part of the cone between two meridian planes which make an angle of $\theta=180°\div\pi=57°.3-$, then will t_8y_8, t_7y_7 etc., be the total hoop compression of the corresponding rings of the cone. It is to be noticed that this quantity does not change sign in the cone, and is always compression.

The meridian compression is expressed, under the same circumstances by the quantities dt_8, dt_7, etc.

Such a cone as this must be placed upon a cylindrical drum or other support which can exert a resistance in the direction *aa*, but if this support is very slightly displaced by the horizontal radial thrust, a hoop tension will be induced at the base of the cone. As this displacement is very likely to occur it is far better to have the base of the cone sufficiently strong to withstand this tension, which is t_8u_8 when du_8 is the weight of 57°.3 : then the supports will sustain a vertical force alone.

This discussion applies equally well to a cone formed of a network of rings and inclined posts with intermediate panels of glass or other material.

CONICAL DOME OF MASONRY.

Let us assume that the uniform horizontal thickness of the dome to be treated, is one sixteenth of the internal diameter of the base, or one eighth of the internal radius, as shown in Fig. 19. The actual thickness is less than this, but since the horizontal thickness is a convenient quantity, we shall call it the thickness unless otherwise specified.

Pass equidistant horizontal planes as previously stated: then the volumes of these rings may be found by the prismoidal formula. The volume

$$=\tfrac{1}{6}\pi h\,[r_1^2-r_1'^2+4(r^2-r'^2)+r_2^2-r_2'^2],$$

in which h is the height of the ring, r_1 and r_1' are the radii external and internal of one base, r_2 and r_2' of the other, and r and r' of the middle section. Now $r_1-r_1'=r-r'=r_2-r_2'=t$ the thickness of the cone; and

$$r_1+r_2=2r,\ r_1'+r_2'=2r'$$
$$\therefore \text{Volume} = \pi ht(r+r')=2\pi ht\bar{r}$$

when $\frac{1}{2}(r+r')=\bar{r}$ the mean radius of the middle section. From this it is seen that the weights vary in the same manner, and are represented by the same quantities as previously stated in case of a thin cone. Assume that the centers of gravity of any thin lunes cut from these rings by meridian planes making a small angle with each other, are at the middle points a_8, a_7, etc., this assumption is sufficiently exact for the part of the cone near the base, which we are now specially to investigate.

By means of the weights $w_8w_7=u_8u_7$, etc., at some assumed distance from the pole *d*, describe the equilibrium polygon *c*, starting from *n* at the inner third of the base.

Now if the cone stands upon a drum which necessarily exerts a sufficient radial thrust to keep the meridian joints of the cone closed down to the base, then all the circumstances will be precisely as before explained in respect to the metallic dome: but if the drum exerts a less radial thrust, the meridian joints will open near the base, and the conditions of stability of that part of the cone will need to be investigated, as was done in the spherical dome of masonry, by considering the upper part of the dome as sustained by a series of stone arches. From *f* draw fc_7 tangent to the curve *c*; then must c_7b_7 be elongated to m_7b_7 and the other ordinates of *c* must be elongated in the same ratio in order that the equilibrium polygon may be tangent to the exterior limit *fm;* and, further, *fm* and fc_7 are the ratio lines by which to effect the elongation. To find how much the thrust is diminished, draw through the intersection of *fm* with *bd*, a line parallel to fc_7 intersecting the weight line at *w*, and then *v* the point where the horizontal through *w* intersects *fm* gives us the new position of the weight line, and its distance from the pole *d*. This vertical intersects *tt* about midway between t_6 and t_7, thus showing that the meridian joints of the cone will be open from the base to about the point g_6. It is unnecessary to draw the equilibrium polygon in its new position.

We thus obtain the least horizontal thrust against which the dome can stand. The actual thrust which the drum exerts may have any value greater than this least thrust.

It is seen that the effect of diminishing the thickness of the cone, is to carry the tangent point c_7 and the point of no compression nearer to the base. In other words the thin dome of masonry of given semi-vertical angle necessarily exerts a greater thrust in proportion to its weight than does a thick dome, though that proportion is unchanged if the joints are to remain closed all the way to the base.

All of the circumstances respecting radial thrust above the point of no hoop compression, and respecting meridian thrust, are the same as in the metallic cone.

Any additional loading above that of the weight of the cone itself, as for example, the weight of a lantern, or of an external dome, as in the case of St. Paul's, can be introduced and treated as an additional height or thickness of certain rings of the cone. The same method which has been here applied may be applied to all such cases, if the weights be determined by some suitable process. For example, it may be shown by the help of the prismoidal formula, that the volume of the ring cut from a uniformly tapering cone by equidistant horizontal planes, varies as the product of the mean radius of the mid-section by the thickness at the mid-section.

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